ON IWASAWA THEORY OVER $\mathbb{Z}[[\mathbb{Z}_p]]$

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Abstract. We establish several abstract ring-theoretical properties of $\mathbb{Z}[[\mathbb{Z}_p]]$ and then apply these results to the study of inverse limits of arithmetic modules in $\mathbb{Z}_p$-extensions of global fields. This approach has a variety of concrete consequences, including the following: we obtain results on the $\mathbb{Z}[[\mathbb{Z}_p]]$-structure of class groups in $\mathbb{Z}_p$-extensions that suggest a strengthened version 'over $\mathbb{Z}$' of the Gross-Kuz'min Conjecture entailing new restrictions on the growth of class groups in cyclotomic $\mathbb{Z}_p$-extensions of general number fields; we prove precise links between Stickelberger elements and Fitting ideals over $\mathbb{Z}[[\mathbb{Z}_p]]$ of dual Selmer groups of $\mathbb{G}_m$ over constant $\mathbb{Z}_p$-extensions of global function fields, respectively cyclotomic $\mathbb{Z}_p$-extensions of CM extensions of totally-real fields, that suggest a 'main conjecture of Iwasawa theory over $\mathbb{Z}[[\mathbb{Z}_p]]$' in these settings; we shed light on a problem of Kato concerning universal norms of unit groups.

Contents

1. Introduction

For a ring A and rational prime p, the completed group ring $A[[\mathbb{Z}_p]]$ is the inverse limit $\varprojlim_n A[\mathbb{Z}/(p^n)]$, where the transition morphisms $A[\mathbb{Z}/(p^{n+1})] \to A[\mathbb{Z}/(p^n)]$ are the group ring maps induced by the natural projections $\mathbb{Z}/(p^{n+1}) \to \mathbb{Z}/(p^n)$.

Arithmetic modules over $\mathbb{Z}[[\mathbb{Z}_p]]$ arise naturally from the limits of families of modules in $\mathbb{Z}_p$-towers of fields. However, since $\mathbb{Z}[[\mathbb{Z}_p]]$ is neither Noetherian nor compact, such modules are usually considered by passing to pro-p completions and working over the associated Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p]]$.

Nevertheless, the passage to pro-p completion can lose significant information and previous authors have either stressed the benefits, and difficulties, in concrete situations of working over $\mathbb{Z}[[\mathbb{Z}_p]]$ or $\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ for a prime $\ell \neq p$ rather than $\mathbb{Z}_p[[\mathbb{Z}_p]]$, or made efforts to understand particular aspects of the problem. This is the case, for example, in the articles of Washington [46, §VI] where it is pointed out that very little is known about the structure of modules over $\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ for $\ell \neq p$ (and see also Lemma 3.1 where it is proved that the Krull dimension of $\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ is infinite), and of Coleman [6] where analogues for $\mathbb{Z}[[\mathbb{Z}_p]]$ of the Weierstrass Preparation Theorem play a key role in the characterization of norm compatible families of units.

With these general problems in mind, we shall here make tentative steps towards the development of a workable theory of $\mathbb{Z}[[\mathbb{Z}_p]]$-modules. Our approach relies on a detailed analysis of several abstract ring-theoretical properties of $\mathbb{Z}[[\mathbb{Z}_p]]$ and for this we use concepts from the theory of non-noetherian rings. In particular, to state our main result in this direction, we use the notions of 'finite conductor rings', 'weak Krull dimension', 'strong n-coherence' and 'strict (n, d)-domains' (for suitable non-negative integers n and d) that are recalled in §2.5.

In this result we shall also, for convenience, refer to a prime number p as 'exceptional' if it is simultaneously irregular, validates Vandiver's Conjecture and is such that the p-adic λ-invariant of every (odd) isotypic factor of the ideal class group of $\mathbb{Q}(e^{2\pi i/p})$ is at least $p-1$. (We note that a probabilistic argument of Washington [33, Chap. 10, App.] suggests there are only finitely many exceptional primes and, more concretely, that the computations of Hart, Wilson and Ong [19] show there are no exceptional primes up to 2^{31}).

Theorem 1.1. *The completed group ring $\mathbb{Z}[[\mathbb{Z}_p]]$ is not a finite conductor ring. However, it is a strongly 2-coherent domain of weak Krull dimension two and, if p is not exceptional, then it is also a strict $(2, 2)$-domain.*

The final assertion of this result implies that, if p is not exceptional, then $\mathbb{Z}[[\mathbb{Z}_p]]$ provides an example of the rings explicitly sought by Costa in [7, §7]. We mention this since, whilst the fact that such rings exist is not new, $\mathbb{Z}[[\mathbb{Z}_p]]$ is both more natural, and much more elementary, than previously known examples that have all involved either power series rings over valuation domains of rank greater than one (as in [7, Ex. 4.4]) or variations of the classical '$D + M$'-construction (as used, for example, in [8]).

The proof of Theorem 1.1 will be given in §2.5 and relies on a detailed study of the category of 'pro-discrete' $\mathbb{Z}[[\mathbb{Z}_p]]$-modules that we introduce in §2.2, including the proof of natural analogues of Nakayama's Lemma and Roiter's Lemma and a careful analysis of finite presentability in this setting, together with an explicit description of the finitely generated prime spectrum of $\mathbb{Z}[[\mathbb{Z}_p]]$.

Theorem 1.1 implies that the ring $\mathbb{Z}[[\mathbb{Z}_p]]$ is not coherent (since it does not have the finite conductor property), and hence falls outside the scope of many of the general techniques discussed by Glaz [15], and also leaves open important questions about the algebraic properties of $\mathbb{Z}[[\mathbb{Z}_p]]$ (cf. Remark 2.22). Nevertheless, it establishes that $\mathbb{Z}[[\mathbb{Z}_p]]$ has sufficiently nice properties to allow for some interesting arithmetic applications.

To give a concrete example, for each number field, respectively global function field, K, each set S of places of K containing the set $S_\infty(K)$ of archimedean places and each finite extension E of K, we write S_E for the set of places of E lying above those in S and

$\mathrm{Cl}_S(E)$ for the S_E-ideal class group, respectively degree zero S_E-divisor class group, of E and we abbreviate $\mathrm{Cl}_S(E)$ to $\mathrm{Cl}(E)$ if $S = S_\infty(K)$. We also fix a $\mathbb{Z}_p$-extension K_∞ of K and, for each natural number n, write K_n for the unique extension of K in K_∞ of degree p^n and then denote the inverse limit over n of the groups $\mathrm{Cl}_S(K_n)$ (with respect to the natural norm maps) by $\mathrm{Cl}_S(K_\infty)$. For each prime ℓ we write $\mathrm{Cl}_S(K_n)_\ell$ and $\mathrm{Cl}_S(K_\infty)_\ell$ for the maximal pro-ℓ subgroups of $\mathrm{Cl}_S(K_n)$ and $\mathrm{Cl}_S(K_\infty)$. We also note that, if $\ell \neq p$, then $\mathrm{Cl}_S(K_{n-1})_\ell$ identifies with a subgroup of $\mathrm{Cl}_S(K_n)_\ell$ and we write $\mathrm{Cl}_S(K_n, K_{n-1})_\ell$ for the quotient $\mathrm{Cl}_S(K_n)_\ell/\mathrm{Cl}_S(K_{n-1})_\ell$. For each such ℓ we also write $\mathrm{ord}_n(\ell)$ for the multiplicative order of ℓ in the ring of residues modulo p^n and define the ℓ-rank $\mathrm{rk}_\ell(M)$ of a finitely generated abelian group M via the equality $|M/(\ell M)| = \ell^{\mathrm{rk}_\ell(M)}$.

Finally, we fix an identification of $\mathrm{Gal}(K_\infty/K)$ with $\mathbb{Z}_p$, and thereby regard $\mathrm{Cl}(K_\infty)$ as a module over $\mathbb{Z}[[\mathbb{Z}_p]]$, and we say that this module is 'finitely ∞-presented' if it is finitely n-presented for every natural number n.

Then the algebraic methods that are developed to prove Theorem 1.1 allow us (in §4.1) to prove the following result.

Theorem 1.2. *For each $\mathbb{Z}_p$-extension K_∞/K of global fields, and set of places S of K containing $S_\infty(K)$, the $\mathbb{Z}[[\mathbb{Z}_p]]$-module $\mathrm{Cl}_S(K_\infty)$ has the following properties.*

- (i) *If K is a function field and K_∞ is the constant $\mathbb{Z}_p$-extension K_∞^{con}, then $\mathrm{Cl}_S(K_\infty)$ is pro-discrete, torsion and finitely ∞-presented.*
- (ii) *If K is a number field, then the following claims are valid.*
  - (a) *$\mathrm{Cl}_S(K_\infty)$ is pro-discrete and torsion.*
  - (b) *$\mathrm{Cl}_S(K_\infty)$ is finitely ∞-presented if and only if both of the following conditions are satisfied:*
    - (i) *for every n, the module $\mathrm{Cl}_S(K_\infty)_p^{\mathrm{Gal}(K_\infty/K_n)}$ is finite;*
    - (ii) *as ℓ ranges over primes with $\ell \neq p$, the quantities*
$$\mathrm{ord}_n(\ell)^{-1} \cdot \mathrm{rk}_\ell\big(\mathrm{Cl}_S(K_n, K_{n-1})_\ell\big)$$
      are bounded independently of both n and ℓ.

If K_∞ is the cyclotomic $\mathbb{Z}_p$-extension K_∞^{cyc} of a number field K and S is the set $S(p)$ comprising all places that are either archimedean or p-adic (in which case $\mathrm{Cl}_S(K_n^{\mathrm{cyc}})$ is the 'p-ideal class group' of K_n^{cyc}), then the validity of the condition in claim (ii)(b)(i) is a well-known generalization of the conjecture of Gross and Kuz'min (for details see Remark 4.1). This observation combines with the result of claim (i) of Theorem 1.2 and the general expectation that ideal class groups over the cyclotomic $\mathbb{Z}_p$-extensions of number fields should behave similarly to degree zero divisor class groups over constant $\mathbb{Z}_p$-extensions of global function fields, to suggest the following refined version 'over $\mathbb{Z}$' of the generalized Gross-Kuz'min Conjecture.

Conjecture 1.3. *For each number field K the $\mathbb{Z}[[\mathbb{Z}_p]]$-module $\mathrm{Cl}_{S(p)}(K_\infty^{\mathrm{cyc}})$ is finitely ∞-presented and, for each $\ell \neq p$, the growth bounds on $\mathrm{Cl}_{S(p)}(K_n^{\mathrm{cyc}}, K_{n-1}^{\mathrm{cyc}})_\ell$ in Theorem 1.2(ii) are valid.*

Remark 1.4. If K is an abelian field, then the condition in Theorem 1.2(ii)(b)(i) is valid for $S = S(p)$ (by Greenberg [17]) and for each prime $\ell \neq p$, the group $\mathrm{Cl}(K_\infty^{\mathrm{cyc}})_\ell$ is finite (by

Washington [47]) and so the condition in Theorem 1.2(ii)(b)(ii) is automatically satisfied in any case in which only finitely many primes divide the orders of $\mathrm{Cl}(K_n^{\mathrm{cyc}})$ as n varies. In this case, Horie [21, Th. 2] has also shown that $\mathrm{Cl}(K_\infty^{\mathrm{cyc}})_\ell$ is trivial for a set of primes ℓ of analytic density one (but see also Remark 1.5 below). More generally, Conjecture 1.3 would appear to be consistent with Cohen-Lenstra-type heuristics since the number of automorphisms of a finite abelian group of given ℓ-power order grows exponentially with its ℓ-rank. Indeed, for certain special classes of abelian fields K, such heuristic arguments have already led to predicted growth bounds that are much stronger than Conjecture 1.3: see, for example, the conjectures of Buhler, Pomerance and Robertson [4] and of Miller [35] concerning the class numbers of real cyclotomic fields (and also Remark 1.6 below in this regard).

Remark 1.5. Since $\mathbb{Z}[[\mathbb{Z}_p]]$ is not Noetherian, the finite generation of the $\mathbb{Z}[[\mathbb{Z}_p]]$-module $\mathrm{Cl}_S(K_\infty)$ does not imply that the submodule given by the direct sum $\bigoplus_\ell \mathrm{Cl}_S(K_\infty)_\ell$ over all primes ℓ is finitely generated. In particular, the latter module cannot be finitely generated if the group $\mathrm{Cl}_S(K_\infty)_\ell$ is non-zero for infinitely many ℓ and this can occur, for example, if $S = S_\infty(K)$ and either K is a function field and $K_\infty = K_\infty^{\mathrm{con}}$ (cf. Rosen [37, Cor. to Th. 11.6]) or K is an imaginary abelian number field and $K_\infty = K_\infty^{\mathrm{cyc}}$ (cf. Washington [46, §III, Cor. 3]).

Remark 1.6. If K is totally real and K_n^{cyc} validates Leopoldt's Conjecture, then the $\mathrm{Gal}(K_\infty^{\mathrm{cyc}}/K_n^{\mathrm{cyc}})$-invariants of $\mathrm{Cl}(K_\infty^{\mathrm{cyc}})_p$, and hence also of $\mathrm{Cl}_{S(p)}(K_\infty^{\mathrm{cyc}})_p$, are finite (this observation is due originally to Iwasawa in his unpublished work [23, §4] and also follows from the results of Kolster in [29, Cor. 1.16, Th. 1.19]). For totally real fields K, therefore, the same reasoning as above would suggest it is reasonable to expect that the predictions of Conjecture 1.3 are valid for ideal class groups rather than just p-ideal class groups.

The algebraic techniques developed to prove Theorems 1.1 and 1.2 can also be used to derive arithmetic consequences going beyond the study of ideal class groups.

In the present article, for example, we combine this approach with the recent verification by Dasgupta and Kakde [10] of a conjecture of Kurihara from [30] to give an unconditional proof of a precise link between Stickelberger elements and the integral dual Selmer groups of $\mathbb{G}_m$ over $\mathbb{Z}_p$-extensions in the setting of both global function fields and CM extensions of totally-real fields (see Theorem 4.5), and also use the approach to shed light on a question about universal norms of p-units of number fields raised by Kato in [26] (see §4.3).

We note, in particular, that the result of Theorem 4.5 suggests a natural 'main conjecture of Iwasawa theory over $\mathbb{Z}[[\mathbb{Z}_p]]$' (see Question 4.7) and hence also motivates the development of a general structure theory for finitely presented torsion $\mathbb{Z}[[\mathbb{Z}_p]]$-modules as a refinement of the classical structure theory for Iwasawa modules. We hope to consider these problems further elsewhere.

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2. Ring-theoretic properties of $\mathbb{Z}[[\mathbb{Z}_p]]$

In this section we investigate the basic ring-theoretic properties of $\mathbb{Z}[[\mathbb{Z}_p]]$. In particular, after carefully studying the category of 'pro-discrete modules' and classifying all finitely generated prime ideals of $\mathbb{Z}[[\mathbb{Z}_p]]$, we shall prove Theorem 1.1 in §2.5.

2.1. **Preliminaries concerning augmentation ideals.** We fix a finite set of rational primes $\mathcal{V}$ that does not contain p, and define a subring of $\mathbb{Q}$ by setting

$$\mathbb{Z}_{\mathcal{V}} := \mathbb{Z}[1/\ell : \ell \in \mathcal{V}]$$

(so that $\mathbb{Z}_{\varnothing} = \mathbb{Z}$). We also set

$$\Gamma := \mathbb{Z}_p \quad \text{and} \quad \Gamma_n := \mathbb{Z}/(p^n) \text{ for each non-negative integer } n,$$

and then define algebras

$$R_n := \mathbb{Z}_{\mathcal{V}}[\Gamma_n] \quad \text{and} \quad R := \mathbb{Z}_{\mathcal{V}}[[\Gamma]] = \varprojlim_n R_n,$$

where the limit is taken with respect to the natural projection maps $R_n \to R_m$ for $n \geq m$.

For each n we write ϱ_n for the projection map $R \to R_n$. We set

$$\varpi_n := \gamma^{p^n} - 1 \in R$$

and consider the associated 'augmentation' ideals of R

$$J_n := R \cdot \varpi_n \subseteq I_n := \ker(\varrho_n).$$

For convenience, we also set $I := I_0$ and $J := J_0 = R \cdot \omega_0$.

The following technical result concerning these ideals will play an important role in the sequel.

Proposition 2.1. *For each non-negative integer n the following claims are valid.*

(i) *The R-module J_n is free of rank one and there exists a natural short exact sequence*

$$0 \to R \xrightarrow{\times \varpi_n} R \to R/J_n \to 0.$$

(ii) *There exists a canonical exact sequence of R-modules*

$$0 \to J_n \to I_n \xrightarrow{\kappa_n} \big(\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}\big) \otimes_{\mathbb{Z}_{\mathcal{V}}} R_n \to 0,$$

where R acts on the third term via the homomorphism $1 \otimes_R \varrho_n$ In particular, any generating set for the R-module I_n is uncountable.

(iii) *There exists a canonical exact sequence of R-modules*

$$0 \to \big(\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}\big) \otimes_{\mathbb{Z}_{\mathcal{V}}} R_n \to R/J_n \to R_n \to 0$$

in which R acts on the second term via the homomorphism $1 \otimes_R \varrho_n$ and the third arrow is induced by ϱ_n.

(iv) *For each natural number m one has $I_n + mR = J_n + mR$ and $I_n^m = J_n^{m-1} \cdot I_n$.*

(v) *For each natural number m the R-module I_n^m/I_n^{m+1} is canonically isomorphic to $\mathbb{Z}_p \otimes_{\mathbb{Z}_{\mathcal{V}}} R_n$ (with its natural action of R).*

Proof. To prove claim (i) it is enough to show that ϖ_n is a non-zero divisor in R. This is true since ϖ_n is a non-zero divisor in the pro-p completion of R (which identifies with the standard p-adic Iwasawa algebra of Γ since $\mathcal{V}$ does not contain p).

To prove claim (ii), for each $m > n$, we write $I_{m,n}$ for the kernel $\varpi_n R_m$ of the projection map $R_m \to R_n$ and $T_{m,n}$ for the element $\varrho_m\big(\sum_{i=0}^{i=p^{m-n}-1}\gamma^{p^n i}\big)$ of R_m.

Then for each $m > n$ there is an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_n & \xrightarrow{1\mapsto T_{m+1,n}} & R_{m+1} & \xrightarrow{1\mapsto \varrho_{m+1}(\varpi_n)} & I_{m+1,n} & \longrightarrow & 0 \\ & & \times p\big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & R_n & \xrightarrow{1\mapsto T_{m,n}} & R_m & \xrightarrow{1\mapsto \varrho_m(\varpi_n)} & I_{m,n} & \longrightarrow & 0 \end{array}$$

in which the unlabelled vertical arrows are the natural (surjective) projection maps. Upon passing to the limit over m of such diagrams we therefore obtain an exact sequence of R-modules

$$0 \to R \xrightarrow{1\mapsto\varpi_n} I_n \to \varprojlim\nolimits^1_{m>n}\big(R_n, \times p\big) \to 0. \tag{1}$$

In addition, by passing to the limit over $a > 0$ of the exact commutative diagrams (in which all unlabelled arrows are the natural projection maps)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_n & \xrightarrow{\times p^{a+1}} & R_n & \longrightarrow & \mathbb{Z}_{\mathcal{V}}/(p^{a+1})\otimes_{\mathbb{Z}_{\mathcal{V}}} R_n & \longrightarrow & 0 \\ & & \times p\big\downarrow & & \mathrm{id}\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & R_n & \xrightarrow{\times p^{a}} & R_n & \longrightarrow & \mathbb{Z}_{\mathcal{V}}/(p^{a})\otimes_{\mathbb{Z}_{\mathcal{V}}} R_n & \longrightarrow & 0 \end{array} \tag{2}$$

one finds that the connecting homomorphism in the long exact sequence of derived limits induces a canonical isomorphism of R-modules

$$\varprojlim\nolimits^1_{m>n}\big(R_n, \times p\big) \cong (\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}})\otimes_{\mathbb{Z}_{\mathcal{V}}} R_n, \tag{3}$$

where R acts on the tensor product via the homomorphism $1\otimes_R \varrho_n$. This isomorphism combines with the exact sequence (1), and the fact that $J_n = R\cdot\varpi_n$, to give an exact sequence as in claim (ii). This sequence then directly implies the final assertion of claim (ii) since R_n is countable but $\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}$ is uncountable.

We also note that, by explicitly computing the connecting homomorphism in the long exact sequence of derived limits induced by the diagram (2), one can construct a family of right inverses $\tilde{\kappa}_n$ to the function κ_n in claim (ii) as follows.

Fix elements

$$a = \sum_{i\geq 0} p^i a_i \in \mathbb{Z}_p\otimes_{\mathbb{Z}_{\mathcal{V}}} R_n \quad\text{and}\quad c\in R_n,$$

where in the first expression each a_i belongs to R_n. For each $m > n$, choose an element a'_{m-1-n} of R_{m-1} that maps to a_{m-1-n} in R_n; set $c_n := c$ and, for each $m > n$, inductively fix an element c_m of R_m that maps to $c_{m-1} + a'_{m-1-n}T_{m-1,n}$ in R_{m-1}. It is then clear that

$$\tilde{\kappa}_n(a) = \tilde{\kappa}_n(a, (c_m)_{m>n}) := (\varpi_n\cdot c_m)_{m>n} \tag{4}$$

belongs to I_n, and an explicit computation of the connecting homomorphism from $\mathbb{Z}_p\otimes_{\mathbb{Z}_\mathcal{V}} R_n$ to the module (3) that arises from the inverse limit (over n) of the diagram (2) shows that

$$\kappa_n(\tilde{\kappa}_n(a)) = \nu_n(a), \tag{5}$$

where ν_n is the natural map $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n \to (\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} R_n$. (For more on the maps $\tilde{\kappa}_n$ see Remark 2.2 below).

The exact sequence in claim (iii) is derived from that in claim (ii) by applying the kernel-cokernel sequence to the sequence of inclusions $J_n \subset I_n \subset R$.

The first equality in claim (iv) is immediate if m is divisible only by primes in $\mathcal{V}$, as it is then a unit in R. On the other hand, if m is divisible by a prime not in $\mathcal{V}$ then, since the group $\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}$ is divisible, the claim follows easily upon applying the Snake Lemma to the commutative diagram obtained by applying the 'multiplication by m' morphism to the short exact sequence in claim (iii).

To prove the second equality in claim (iv) it is enough to show that $I_n^2 = J_nI_n$ since, if this is true, then for $m > 2$ one has $I_n^m = I_n^2 I_n^{m-2} = J_n I_n^{m-1}$.

In a similar way, if claim (iv) is true then for each $m > 1$, multiplication by ϖ_n^{m-1} induces an isomorphism of R-modules $I_n/I_n^2 \cong I_n^m/I_n^{m-1}$, and so claim (v) is reduced to the claim that I_n/I_n^2 is canonically isomorphic to $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$.

Since the exact sequence in claim (ii) implies that $I_n^2 \subseteq J_n$, to complete the proof of both claims (iv) and (v) it is thus enough to show that there exists a canonical isomorphism $I_n/I_n^2 \to \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$ of R-modules that maps the submodule J_n/I_n^2 onto $R_n \cong R/I_n$.

To show this we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_n/I_n^2 & \xrightarrow{\subset} & I_n/I_n^2 & \xrightarrow{\kappa_n} & (\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} R_n & \longrightarrow & 0 \\
 & & & & {\scriptstyle\theta_n}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & R_n & \xrightarrow{\subset} & \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n & \longrightarrow & (\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} R_n & \longrightarrow & 0.
\end{array} \tag{6}$$

In this diagram the upper row is the exact sequence induced by that in claim (ii), the lower row is the tautological exact sequence and θ_n denotes the following composite homomorphism of R-modules

$$I_n/I_n^2 \to \varprojlim_{m>n} I_{m,n}/I_{m,n}^2 \xrightarrow{\sim} \varprojlim_{m>n} (\mathbb{Z}/(p^{m-n}) \otimes_{\mathbb{Z}_\mathcal{V}} R_n) = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n.$$

Here the first map is the natural projection and, writing $I'_{m,n}$ for the augmentation ideal of the group ring of the subgroup of Γ_m generated by the image $\gamma_{m,n}$ of γ^{p^n}, the second is induced by the canonical isomorphisms

$$\frac{I_{m,n}}{I_{m,n}^2} = \frac{R_m I'_{m,n}}{(R_m I'_{m,n})^2} \cong \frac{I'_{m,n}}{(I'_{m,n})^2} \otimes_{\mathbb{Z}_\mathcal{V}} R_n \cong \langle \gamma_{m,n} \rangle \otimes_{\mathbb{Z}_\mathcal{V}} R_n \cong (\mathbb{Z}/(p^{m-n})) \otimes_{\mathbb{Z}_\mathcal{V}} R_n$$

in which the third map is induced by the canonical group isomorphism $I'_{m,n}/(I'_{m,n})^2 \cong \langle \gamma_{m,n} \rangle$ and the fourth by sending each element $\gamma_{m,n}^t$ to the residue class of t modulo p^{m-n}.

By explicit computation one checks that, for each x in R and a in $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$, the map θ_n sends the image of $x\varpi_n$ in J_n/I_n^2 to the image of x in R_n and the image of $\tilde{\kappa}_n(a)$ in I_n/I_n^2 to a. In particular, since $\theta_n\big(J_n/I_n^2\big) = R_n$, it is enough for us to show that θ_n is bijective.

Now, since the equality (5) combines with the exact sequence in claim (ii) to imply that every element of I_n is equal to $x\varpi_n + \tilde{\kappa}_n(a)$ for some x in R and a in $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$, it also combines with the equality $\theta_n\big(J_n/I_n^2\big) = R_n$ to imply that (6) commutes. By applying the Snake Lemma to this diagram, it is therefore enough to show θ_n is injective. However, if $x = (x_m)_{m>n}$ is any element of the submodule I_n of $R = \varprojlim_{m>n} R_m$ with the property that $x_m \in I_{m,n}^2$ for all $m > n$, then for each m one has $x_m = (\gamma - 1)x'_m$ for some element x'_m of $I_{m,n}$ and so it is enough to show that $(x'_m)_{m>n}$ belongs to $I_n = \varprojlim_{m>n} I_{m,n}$. However, this is true since, writing π_m for the natural projection $R_m \to R_{m-1}$, the element $\pi_m(x'_m) - x'_{m-1}$ belongs to both $I_{m-1,n}$ and the kernel $R \cdot T_{m-1,n}$ of multiplication by ϖ_n on R_{m-1}. Since $I_{m-1,n} \cap R \cdot T_{m-1,n} = \{0\}$, one therefore has $\pi_m(x'_m) = x'_{m-1}$ for all $m > n$, and hence that $(x'_m)_{m>n}$ belongs to I_n, as required. □

Remark 2.2. For any family $\tilde{c} := (c_m)_{m>n}$ constructed as in (4), the map $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n \to I_n$ that sends a to $\tilde{\kappa}_n(a, \tilde{c})$ is not a homomorphism of R-modules. In addition, whilst the exact sequence in Proposition 2.1(ii) implies $\tilde{\kappa}_n(a, \tilde{c})$ need not be divisible by ϖ_n in R, it is always divisible by ϖ_n in the classical Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p]]$ and the corresponding quotient can be computed explicitly as follows. Set $b_n := a$ and then, for each $m > n$, inductively set $b_m := p^{-1}(b_{m-1} - a_{m-1-n}) \in \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$; choose an element b'_m of $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_m$ that maps to b_m in $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$ and then finally set $x_m := c_m + b'_m \cdot T_{m,n} \in \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_m$. One then checks that the family $x := (x_m)_{m>n}$ defines an element of $\Lambda = \varprojlim_{m>n}(\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_m)$ with the property that $\tilde{\kappa}_n(a, \tilde{c}) = x \cdot \varpi_n$.

Remark 2.3. The fact that $J \neq I$ causes significant differences between the properties of modules over R and over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$, such as the following.
(i) Proposition 2.1(ii) (with $n = 0$ and $\mathcal{V} = \varnothing$) implies that the trivial $\mathbb{Z}[[\mathbb{Z}_p]]$-module $\mathbb{Z}$ is not finitely presented.
(ii) Whilst $\mathsf{Tor}^1_\Lambda(\mathbb{Z}_p, M)$ vanishes for any ideal M of Λ (cf. [36, Prop. (5.5.3)(iv)]), the results of Proposition 2.1(i), (ii) and (v) (each with $n = 0$) combine to imply that $\mathsf{Tor}^1_R(\mathbb{Z}_\mathcal{V}, I)$ identifies with the (uncountably infinite dimensional) rational vector space $(\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} \mathbb{Z}_p$.

Remark 2.4. It seems to be expected that the completion of a coherent ring at a finitely generated ideal is flat (this result is, for example, stated as [20, Th. 3.58] and has been used elsewhere in the literature, but relies on a proof of the Artin-Rees Lemma for coherent modules over coherent rings in [20, Cor. 3.55] that can be seen to be incomplete and appears difficult to fix). However, since we will later show that R is not coherent, one cannot a priori expect it to have the same property and, in fact, the completion $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$ of R at the ideal p fails to be flat even on the category of cyclic modules. To see this, fix an element x of I whose image in $\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}$ (under the homomorphism κ in the exact sequence of Proposition 2.1(ii)) has infinite order. Then, in this case, the map $\lambda : \mathbb{Z}_\mathcal{V} \to R/J$ sending 1 to the class of x defines an injective map of cyclic R-modules. However, since $(R/J) \otimes_R \Lambda = \Lambda/(J \cdot \Lambda)$, and $x \in I \cdot \Lambda = J \cdot \Lambda$, the map $\lambda \otimes_R \Lambda$ is zero.

2.2. **The category of pro-discrete modules.** In this section we introduce an important category of 'pro-discrete' $\mathbb{Z}_\mathcal{V}[[\mathbb{Z}_p]]$-modules and then prove natural analogues of both Nakayama's Lemma and Roiter's Lemma for this category.

To do this we continue to use the notation introduced at the beginning of §2.1 (so that R denotes $\mathbb{Z}_\mathcal{V}[[\mathbb{Z}_p]]$, R_n denotes $\mathbb{Z}_\mathcal{V}[\Gamma_n]$ etc.)

2.2.1. *The definition and basic properties of pro-discrete modules.* For each R-module M and non-negative integer n we set

$$M_{[n]} := (R/J_n) \otimes_R M \cong M/\varpi_n(M) \ \text{ and } \ M_{(n)} := (R/I_n) \otimes_R M \cong R_n \otimes_R M.$$

The following result records some useful relations between these coinvariant modules.

Lemma 2.5. *For each R-module M and each non-negative integer n the following claims are valid.*

(i) *For any R-module M, there exists a canonical exact sequence of R-modules*

$$M^{\varpi_n=0} \to \ker\big(I_n \otimes_R M \to M\big) \to (\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} M_{(n)} \xrightarrow{\theta} M/\varpi_n(M) \to M_{(n)} \to 0.$$

In particular, if either every element of $M_{(n)}$ has finite order, or $M/\varpi_n(M)$ has no non-zero divisible subgroup, then $M_{(n)} = M/\varpi_n(M)$.

(ii) *For any short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$, there exists a canonical exact sequence of R-modules*

$$0 \to M_1^{\varpi_n=0} \to M_2^{\varpi_n=0} \to M_3^{\varpi_n=0} \to M_{1,[n]} \to M_{2,[n]} \to M_{3,[n]} \to 0.$$

Proof. For any R-module M, the exact sequence in Proposition 2.1(i) induces an identification of $\mathsf{Tor}^1_R(R_{[n]}, M)$ with $M^{\varpi_n=0}$. In addition, the tautological short exact sequence $0 \to I_n \to R \to R_{(n)} \to 0$ induces an identification of $\mathsf{Tor}^1_R(R_{(n)}, M)$ with the kernel of the natural map $I_n \otimes_R M \to M$. Given these identifications, the exact sequence in claim (i) is obtained by applying the functor $- \otimes_R M$ to the exact sequence in Proposition 2.1(iii).

The final assertion of claim (i) then follows from the displayed exact sequence and the fact that, since $\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}$ is divisible, the stated hypotheses imply that θ is the zero map.

For every R-module M we have an exact sequence

$$0 \to M^{\varpi_n=0} \to M \xrightarrow{\times\varpi_n} M \to M_{[n]} \to 0$$

The exact sequence of claim (ii) now follows by combining this fact (with M equal to each of M_1, M_2 and M_3) with the Snake Lemma. □

We shall now introduce a key concept in our theory.

Definition 2.6. An inverse system $(M_n)_n = (M_n, \pi_n)_n$ of R-modules indexed by non-negative integers n will be said to be a *pro-discrete system* if each M_n is naturally an R_n-module and, for each $n > 0$, the transition morphism π_n induces an isomorphism of R_{n-1}-modules

$$\overline{\pi}_n : R_{n-1} \otimes_R M_n \cong M_{n-1}. \tag{7}$$

An R-module will be said to be *pro-discrete* if it is equal to the limit of a pro-discrete system of R-modules.

Remark 2.7. If M is the limit of a pro-discrete system $(M_n, \pi_n)_n$, then for every $n > 0$ the bijectivity of $\overline{\pi}_n$ implies that the transition morphism π_n is surjective. The natural projection map $M \to M_n$ therefore induces a surjective map of the form $M_{(n)} \to M_n$.

Remark 2.8. The ring R is itself a pro-discrete R-module since it is the limit of the pro-discrete system $(R_n, \varrho_{n,n-1})_n$ in which each $\varrho_{n,n-1}$ is the natural map $R_n \to R_{n-1}$. In general, however, it seems difficult to decide if a given R-module is pro-discrete (at the moment, for example, we do not know whether the augmentation ideal I_0 studied in §2.1 is pro-discrete). Nevertheless, any R-module N gives rise to a pro-discrete system $(N_{(n)}, \pi_n)_n$, with π_n the canonical map $N_{(n)} \to N_{(n-1)}$, and hence to a pro-discrete module $\varprojlim_n N_{(n)}$. In Proposition 2.13(i) below we will show that the canonical map $M \to \varprojlim_n M_{(n)}$ is bijective for every finitely generated pro-discrete R-module M, and this fact implies that many finitely-presented R-modules are not pro-discrete. For instance, the cyclic R-module R/J is not pro-discrete (see Remark 2.14) and, since the modules $J = \varpi_0 R$ and R are pro-discrete, this example shows that the category of pro-discrete modules is not abelian.

Remark 2.9. If H is a finite abelian group, then we can consider the associated group rings $R[H]$ and $R_n[H]$ over R and R_n (so that $R[H] = \varprojlim_n R_n[H]$). If M is a pro-discrete R-module in which each M_n is an $R_n[H]$-module and each π_n is a homomorphisms of $R_n[H]$-modules, then we refer to M as a pro-discrete $R[H]$-module.

We end this section by proving a useful refinement of the observation in Remark 2.7.

Lemma 2.10. *Let M be the limit of a pro-discrete system $(M_n, \pi_n)_n$ in which each M_n is finite and of order prime-to-p. Then, for each m, the given maps $M_n \to M_m$ for $n > m$ induce isomorphisms $M^{\varpi_m=0} \cong M_n^{\varpi_m=0} \cong M_m$ and $M_{(m)} \cong M_m$.*

Proof. Since, for each $n > m$, the R_n-module M_n is finite and of order prime-to-p, it decomposes as a direct sum $M_n^{\varpi_m=0} \oplus \varpi_m(M_n)$. This decomposition combines with the given map $M_n \to M_m$ to induce an identification

$$M_m \cong R_m \otimes_{R_n} M_n = M_n/\varpi_m(M_n) \cong M_n^{\varpi_m=0},$$

and hence an exact sequence $0 \to M_m \to M_n \xrightarrow{\varpi_m} M_n \to M_m \to 0$. Passing to the limit over $n > m$ of these sequences preserves exactness (since each occurring module is finite) and hence gives an exact sequence of R-modules $0 \to M_m \to M \xrightarrow{\varpi_m} M \to M_m \to 0$. This sequence in turn induces the required identifications

$$M^{\varpi_m=0} \cong M_m \cong M/\varpi_m(M) = M_{(m)},$$

where the last equality follows from Lemma 2.5(i) and the fact M_m is finite. □

2.2.2. *Nakayama's Lemma and Roiter's Lemma.* In the sequel, for any finitely generated module M over a ring A we write $\mu_A(M)$ for the minimal number of generators of M. We also recall that an $R_n[H]$-module M is said to be 'locally-free (of rank d)' if for every rational prime ℓ the $R_{n,\ell}[H]$-module $\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} M$ is free (of rank d). We shall also often refer to a module that is locally-free of rank one as 'invertible'.

If G is any finite abelian group and H a subgroup thereof, we shall write e_H for the idempotent $|H|^{-1}\sum_{h\in H} h$ of $\mathbb{Q}[G]$.

The following result provides us with suitable analogues of both Nakayama's Lemma and Roiter's Lemma for the category of pro-discrete modules.

Theorem 2.11. *Let H be a finite abelian group and M a pro-discrete $R[H]$-module. Then the following claims are valid.*

(i) *M is finitely generated if and only if it contains a finite subset which, for every n, projects to give a set of generators of the $R_n[H]$-module M_n.*

In the remaining claims we assume there exists a natural number d such that $\mu_{R_n[H]}(M_n) \le d$ for every n.

(ii) *M is finitely generated and $\mu_{R[H]}(M) \le \mu_{\mathbb{Z}_{\mathcal{V}}[H]}(M_0) + d$. In addition, there exists a finitely generated pro-discrete $R[H]$-submodule M' of M with both of the following properties:*
  (a) *$\mu_{R[H]}(M') \le \mu_{\mathbb{Z}_{\mathcal{V}}[H]}(M_0)$;*
  (b) *For every n, the order of $(M/M')_{(n)}$ is finite and prime-to-p.*

  In particular, if each $R_n[H]$-module M_n is locally-free, then for any natural numbers m and $t > 1$, the module M' can be chosen to be free and such that the order of $\big(M/M'\big)_{(m)}$ is prime to t.

(iii) *Assume $\mathcal{V} = \varnothing$, and H is trivial, so that $R[H] = R = \mathbb{Z}[[\mathbb{Z}_p]]$. Then, unless p is exceptional, the following claims are valid.*
  (a) *If every M_n is a projective R_n-module, then M is a projective R-module.*
  (b) *If every M_n is a free R_n-module of rank d, and the natural projection map $\mathrm{SL}_d(R_n) \to \mathrm{SL}_d(R_{n-1})$ is surjective for every $n > 1$ (which is automatic if $d \neq 2$), then M is a free R-module of rank d.*

Proof. Set $\mathbb{A} := R[H]$ and $\mathbb{A}_n := R_n[H]$ for each n (so that $\mathbb{A}_0 = \mathbb{Z}_{\mathcal{V}}[H]$).

To prove claim (i) it is enough to show that if $\{z_i := (z_{i,n})_n\}_{1\le i\le d}$ is any subset of M with the stated property, then it generates M over $R[H]$. To do this, we consider, for each n, the following exact commutative diagram

$$\begin{array}{ccccccccc} & & & & \ker(\rho_n) & \xrightarrow{\kappa_n'} & \ker(\pi_n) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(\kappa_n) & \longrightarrow & \mathbb{A}_n^d & \xrightarrow{\kappa_n} & M_n & \longrightarrow & 0 \\ & & \rho_n' \downarrow & & \rho_n \downarrow & & \pi_n \downarrow & & \\ 0 & \longrightarrow & \ker(\kappa_{n-1}) & \longrightarrow & \mathbb{A}_{n-1}^d & \xrightarrow{\kappa_{n-1}} & M_{n-1} & \longrightarrow & 0. \end{array} \tag{8}$$

Here κ_n denotes the surjective map of $\mathbb{A}_n$-modules that sends the i-th element in the standard basis of $\mathbb{A}_n^d$ to $z_{i,n}$, ρ_n is the natural (surjective) projection map and ρ_n' and κ_n' are the restrictions of the respective maps ρ_n and κ_n.

We write Δ_n for the subgroup of Γ_n of order p and $I(n)$ for the submodule of $\mathbb{A}_n$ generated by the set $\{h-1 : h \in \Delta_n\}$. Then it is clear that $\ker(\rho_n) = I(\Delta_n)^d$, whilst the isomorphism (7) induced by π_n implies that $\ker(\pi_n) = I(\Delta_n)\cdot M_n$. It follows that the map κ_n' is surjective and hence, by applying the Snake Lemma to the above diagram, that ρ_n' is also surjective so that the derived limit $\varprojlim^1_n \ker(\kappa_n)$ with respect to the maps ρ_n' vanishes. Upon passing to limit over n of the commutative diagrams given by the second and third rows of the above diagram, one therefore deduces that the homomorphism of $\mathbb{A}$-modules $\mathbb{A}^d = \varprojlim_n \mathbb{A}_n^d \to \varprojlim_n M_n = M$ that sends the i-th element in the standard basis of $\mathbb{A}^d$ to z_i is surjective, as required.

To prove claim (ii) we set $N := \mu_{\mathbb{A}_0}(M_0) \le d$ and show first that, for each n, there exists a subset $X_n := \{x_{i,n}\}_{1\le i\le N}$ of M_n with the following two properties:

(P1) the $\mathbb{A}_n$-submodule M'_n of M_n generated by X_n has finite, prime-to-p index;
(P2) for each $n' < n$, the natural map $M_n \to M_{n'}$ sends $x_{i,n}$ to $x_{i,n'}$ for every index i and also induces an isomorphism $R_{n'} \otimes_{R_n} M'_n \cong M'_{n'}$.

To establish this we use induction on n. For the case $n = 0$ the necessary conditions are satisfied by taking X_0 to be any set of generating elements for the $\mathbb{A}_0$-module M_0 (so that $M'_0 = M_0$). For the inductive step we fix $n > 0$ and assume that suitable sets X_m have been constructed for each $m < n$. For each index i with $1 \le i \le N$ we then fix a pre-image $x_{i,n}$ of $x_{i,n-1}$ under the transition map $\pi_n : M_n \to M_{n-1}$, set $X_n := \{x_{i,n}\}_{1\le i\le N}$ and write M'_n for the $\mathbb{A}_n$-submodule of M_n generated by X_n. It is then clear that

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} \pi_n(M'_n) = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M'_{n-1} = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M_{n-1} = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} \pi_n(M_n)$$

and hence that

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M_n = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M'_n + \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} \ker(\pi_n) = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M'_n + I(\Delta_n)(\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M_n).$$

Since $I(\Delta_n)$ belongs to the Jacobson radical of $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} \mathbb{A}_n$, we may therefore deduce from Nakayama's Lemma that $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M_n = \mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} M'_n$. This implies that the index of M'_n in M_n is finite and prime-to-p, and hence that (P1) is satisfied. The first property in (P2) is also clear for this construction, whilst the second property is easily derived from the fact that the natural map $R_{n'} \otimes_{R_n} M'_n \to R_{n'} \otimes_{R_n} M_n \cong M_{n'}$ is injective since the index of M'_n in M_n is prime-to-p.

For each $n > 0$ we now consider the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_n & \xrightarrow{\iota_n} & M_n & \longrightarrow & Q_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi'_n} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \tilde{\pi}_n} & & \\ 0 & \longrightarrow & M'_{n-1} & \xrightarrow{\iota_{n-1}} & M_{n-1} & \longrightarrow & Q_{n-1} & \longrightarrow & 0, \end{array} \tag{9}$$

in which ι_n and ι_{n-1} are inclusion maps, $Q_n := \mathrm{cok}(\iota_n)$, $Q_{n-1} := \mathrm{cok}(\iota_{n-1})$, π'_n is the restriction of π_n and $\tilde{\pi}_n$ is induced by the commutativity of the first square. In particular, since the maps π'_n are surjective, we may pass to the limit over such diagrams to obtain a short exact sequence of $\mathbb{A}$-modules

$$0 \to M' \to M \to Q \to 0, \tag{10}$$

in which we set $M' := \varprojlim_n M'_n$ and $Q := \varprojlim_n Q_n$ (with the limits taken with respect to the maps π'_n and $\tilde{\pi}_n$). In addition, the final assertion of property (P2) implies that M' is a pro-discrete $\mathbb{A}$-module and also combines with claim (i) to imply $\{(x_{i,n})_n\}_{1\le i\le N}$ is a generating set for M' and hence that

$$\mu_{\mathbb{A}}(M') \le N = \mu_{\mathbb{A}_0}(M_0). \tag{11}$$

Next we note that, since M' and M are both pro-discrete, the commutative diagrams (9) can be used to show that, for each $n' < n$, the natural map $R_{n'} \otimes_{R_n} Q_n \to Q_{n'}$ is bijective and hence that Q is pro-discrete. The property in (ii)(b) is therefore true since, for each n, the module $(M/M')_{(n)}$ is isomorphic to $Q_{(n)}$ and hence, by Lemma 2.10, to the module Q_n that is finite and of order prime-to-p.

To establish M is finitely generated and such that $\mu_{\mathbb{A}}(M) \le \mu_{\mathbb{A}_0}(M_0)+d$, we are reduced, via the exact sequence (10) and inequality (11), to showing that $\mu_{\mathbb{A}}(Q) \le d$. Hence, since Q is pro-discrete, claim (i) implies it is enough for us to construct a subset $\{z_i = (z_{i,n})_n\}_{1\le i\le d}$ of Q with the property that, for every n, the $\mathbb{A}_n$-module Q_n is generated by the set $\{z_{i,n}\}_{1\le i\le d}$.

For each $n' \le n$ we write $\Delta(n,n')$ for the subgroup of Γ_n of order $p^{n-n'}$. We then obtain mutually orthogonal idempotents in $R_n[1/p]$ by setting

$$e_{n,n'} := e_{\Delta(n,n'+1)} - e_{\Delta(n,n')} \tag{12}$$

for each $0 < n' < n$, and also $e_{n,0} := e_{\Gamma_n}$. Then there is an equality $\sum_{0\le n'<n} e_{n,n'} = 1$ in $R_n[1/p]$ and hence, since the order of each module Q_m is prime-to-p, Lemma 2.10 implies a corresponding isomorphism of $\mathbb{A}_n[1/p]$-modules

$$Q_n \cong \bigoplus_{0\le n'<n} e_{n,n'}(Q_{n'}). \tag{13}$$

Next we note that, for each n, the surjective map $M_n \to Q_n$ combines with the assumption $\mu_{\mathbb{A}_n}(M_n) \le d$ to imply that we may fix a generating set of the $\mathbb{A}_n$-module Q_n of the form $\{z'_{i,n}\}_{1\le i\le d}$. Then, if, for each index i, we set

$$z_{i,n} := \sum_{0\le n'<n} e_{n,n'} z'_{i,n'},$$

the natural map $Q_n \to Q_{n'}$ sends $z_{i,n}$ to $z_{i,n'}$ and the isomorphism (13) implies that the $\mathbb{A}_n$-module Q_n is generated by $\{z_{i,n}\}_{1\le i\le d}$. It follows that the collection $\{(z_{i,n})_n\}_{1\le i\le d}$ is therefore a set of generators of Q, as required.

To complete the proof of claim (ii) we note that if each $\mathbb{A}_n$-module M_n is locally-free, then the (finite) quantity $d' := \mu_{(\mathbb{Q}\otimes_{\mathbb{Z}_\mathcal{V}}\mathbb{A}_n)}(\mathbb{Q}\otimes_{\mathbb{Z}_\mathcal{V}} M_n)$ is independent of n. Further, by applying Roiter's Lemma (cf. [9, Lem. (31.6)]) over the $\mathbb{Z}_\mathcal{V}$-order $\mathbb{A}_m$, we may choose a free $\mathbb{A}_m$-submodule $M'(m)$ of M_m such that the order of $M_m/M'(m)$ is finite and prime to pt. It follows that the rank of $M'(m)$ is d' and so we can choose a basis $\{x_{i,m}\}_{1\le i\le d'}$. We can then use these elements as the start of an inductive construction, just as above, of a subset $\{x_{i,n}\}_{1\le i\le d'}$ of M_n for each $n \ge m$. The $\mathbb{A}_n$-submodule $M'(n)$ of M_n spanned by this set is then free of rank d' and such that, for each $n \ge n' \ge m$, the natural map $R_{n'} \otimes_{R_n} M'(n) \to M'(n')$ is bijective. The $\mathbb{A}$-submodule $M' := \varprojlim_n M'(n)$ of M is therefore pro-discrete and such that $\big(M/M'\big)_{(m)}$ is isomorphic to $M_m/M'(m)$ and so has order prime to t, as required. It is now enough to show that M' is free of rank d', with basis $\{(x_{i,n})_{n\ge m}\}_{1\le i\le d'}$. This is true since, after replacing M_n by M'_n and $z_{i,n}$ by $x_{i,n}$, each horizontal arrow in the lower right hand commutative square in (8) is bijective and hence, upon passing to the limit over n in these diagrams, one obtains an isomorphism of $\mathbb{A}$-modules of the form $\mathbb{A}^{d'} \cong M'$.

Turning to claim (iii) we assume that $\mathcal{V}$ is the empty set and that H is trivial, so that $\mathbb{A} = R = \mathbb{Z}[[\mathbb{Z}_p]]$. It is convenient to first prove claim (iii)(b) and so we assume that each R_n-module M_n is free of rank d. Then, to prove M is a free R-module of rank d, the above argument reduces us to constructing a subset $\{(z_{i,n})_n\}_{1\le i\le d}$ of M with the property that, for every n, the set $\{z_{i,n}\}_{1\le i\le d}$ is a R_n-basis of M_n.

The construction of suitable sets of elements $\{z_{i,n}\}_{1\le i\le d}$ is then an easy inductive exercise (on n) for any prime p for which the natural projection maps $\mathrm{GL}_d(R_n)\to \mathrm{GL}_d(R_{n-1})$ are surjective for every $n>1$. In particular, if the projection maps $\mathrm{SL}_d(R_n)\to\mathrm{SL}_d(R_{n-1})$ are surjective, then the natural exact sequences

$$1\to \mathrm{SL}_d(R_m)\xrightarrow{\subset}\mathrm{GL}_d(R_m)\xrightarrow{\det} R_m^\times\to 1$$

for $m\in\{n,n-1\}$ reduce us to showing that if p is not exceptional (in the sense defined in the Introduction), then the projection maps $R_n^\times\to R_{n-1}^\times$ are surjective for all $n>1$. But if p is regular, then this is proved by Kervaire and Murthy in [27, Th. 1.3], whilst if p is irregular but not exceptional, then it is proved by Ullom in [42, Cor. 2.6].

To complete the proof of claim (iii)(b), it is thus enough to show that if $d\neq 2$, then the projection maps $\mathrm{SL}_d(R_n)\to\mathrm{SL}_d(R_{n-1})$ are surjective. If $d=1$, then this is obvious (since both groups are trivial) and so we assume that $d>2$ and write $\mathrm{E}_d(R_n)$ for the subgroup of $\mathrm{SL}_d(R_n)$ generated by elementary matrices. Then, since R_n has stable-range 2 (by Bass [9, Th. (40.41)]), the Injective Stability Theorem of Bass and Vaserstein [9, Th. (40.44)] implies that the natural map $\mathrm{SL}_d(R_n)/\mathrm{E}_d(R_n)\to\mathrm{SK}_1(R_n)$ is bijective. In particular, since Alperin, Dennis and Stein [1, Th. 3.4] have shown that $\mathrm{SK}_1(R_n)$ vanishes, one has $\mathrm{SL}_d(R_n)=\mathrm{E}_d(R_n)$. It is therefore enough to note that, for every $n>1$, the definition of elementary matrices combines with the surjectivity of the projection $R_n\to R_{n-1}$ to imply that the natural map $\mathrm{E}_d(R_n)\to\mathrm{E}_d(R_{n-1})$ is surjective.

To prove claim (iii)(a) it now suffices to show that if for a given pro-discrete R-module M every R_n-module M_n is projective, then there exists a pro-discrete R-module $M'=(M'_n)_n$ such that each R_n-module $M_n\oplus M'_n$ is free of rank d for a fixed natural number d with $d>2$. By replacing M by $M\oplus R$, if necessary, we can also assume that the rank of M_n is at least 2 (for every n).

We regard M as fixed and assume that, for some fixed $n\ge 1$, and every $m<n$, there exists an invertible R_m-submodule N_m of $\mathbb{Q}[\Gamma_m]$ such that $N_m=R_m\otimes_{R_{n-1}}N_{n-1}$ and, in addition, $M_m\oplus N_m$ is a free R_m-module whose rank is independent of m and at least three. Then, taking account of the case of free modules considered above, it suffices for us to show the existence of an invertible R_n-submodule N_n of $\mathbb{Q}[\Gamma_n]$ with the property that $M_n\oplus N_n$ is a free R_n-module and also $R_{n-1}\otimes_{R_n}N_n=N_{n-1}$.

As a first step in the construction we recall that, by Swan's Theorem [9, Th. 32.11], M_n is a locally-free R_n-module and so gives rise to a class $[M_n]$ in the locally-free classgroup $\mathrm{Cl}(R_n)$ of the order R_n (as defined in [9, (49.10)]). Following the discussion of [9, just before Rem. (49.11)], we can choose an invertible R_n-submodule N'_n of $\mathbb{Q}[G_n]$ whose class in $\mathrm{Cl}(R_n)$ is the inverse of $[M_n]$. Then, since R_n has locally-free cancellation (cf. the discussion just after [9, Def. (49.29)]), $M_n\oplus N'_n$ is a free R_n-module and the identification

$$R_{n-1}\otimes_{R_n}(M_n\oplus N'_n)=(R_{n-1}\otimes_{R_n}M_n)\oplus(R_{n-1}\otimes_{R_n}N'_n)=M_{n-1}\oplus(R_{n-1}\otimes_{R_n}N'_n)$$

implies that the class of $R_{n-1}\otimes_{R_n}N'_n$ in $\mathrm{Cl}(R_{n-1})$ is equal to the class of N_{n-1}. The Bass-Serre Cancellation Theorem (cf. [48, Prop. 3.4]) therefore implies the existence of an element λ of $\mathbb{Q}[\Gamma_{n-1}]^\times$ such that $N_{n-1}=\lambda\cdot(R_{n-1}\otimes_{R_n}N'_n)$. Fixing a pre-image λ' of λ under the surjective map $\mathbb{Q}[\Gamma_n]^\times\to\mathbb{Q}[\Gamma_{n-1}]^\times$, we therefore obtain an invertible Λ_n-submodule

$N_n := \lambda' \cdot N'_n$ of $\mathbb{Q}[\Gamma_n]$ that is isomorphic to N'_n, so that the R_n-module $M_n \oplus N_n$ is free, and is also such that

$$R_{n-1} \otimes_{R_n} N_n = \lambda \cdot (R_{n-1} \otimes_{R_n} N'_n) = N_{n-1},$$

as required. □

Remark 2.12. It is claimed in [39, Th. 4.1] that the natural homomorphism

$$\mathrm{Pic}(\mathbb{Z}[[\mathbb{Z}_p]]) \to \varprojlim_n \mathrm{Pic}(\mathbb{Z}[\mathbb{Z}/(p^n)])$$

of Picard groups is bijective for every prime p that is both irregular and validates Vandiver's Conjecture. However, this claim is based on a misreading of the result [42, Cor. 2.6] of Ullom that is used in the proof of Theorem 2.11 and the argument of [39] establishes bijectivity of the above map only under the assumption that p is not exceptional.

2.3. **Finite presentability of pro-discrete modules.** In the next result, we establish explicit criteria to guarantee the finite presentability of finitely generated modules.

Proposition 2.13. *Let H be a finite abelian group. Then the following claims are valid for each finitely generated $R[H]$-module M.*

(i) *M is pro-discrete if and only if the natural map $M \to \varprojlim_n M_{(n)}$ is bijective. In particular, if M is the limit of a pro-discrete system $(M_n, \pi_n)_n$ then for every n the natural map $M_{(n)} \to M_n$ is bijective.*

(ii) *If M is pro-discrete, then M is finitely presented if and only if the quantities $\mu_R\big(\mathsf{Tor}^1_R(R_n, M)\big)$ are finite and bounded independently of n. The latter condition is satisfied if, for example, each M_n is finite and the quantities $\mu_R\big(M^{\varpi_n=0}\big)$ are finite and bounded independently of n.*

(iii) *Assume M is a submodule of $R[H]^a$ for some natural number a. Then M is pro-discrete. In addition, M is finitely presented if and only if the quantities*

$$\mu_R\big(\mathsf{Tor}^1_R(R_n, M)\big) = \mu_R\big(\mathsf{Tor}^2_R(R_n, R[H]^a/M)\big)$$

are finite and bounded independently of n. The latter condition is satisfied if, for example, each group $\big(R[H]^a/M\big)^{\varpi_n=0}[p^\infty]$ has bounded exponent.

Proof. Set $\mathbb{A} := R[H]$ and $\mathbb{A}_n := R_n[H]$ for each n.

Then, since M is finitely generated, we can fix an exact sequence of $\mathbb{A}$-modules of the form

$$0 \to K \xrightarrow{\theta} \mathbb{A}^d \to M \to 0. \tag{14}$$

For each pair of natural numbers n and n' with $n > n'$, this sequence gives rise to an exact commutative diagram of $\mathbb{A}_n$-modules

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, M) & \longrightarrow & K_{(n)} & \xrightarrow{\theta_{(n)}} & \mathbb{A}_n^d & \longrightarrow & M_{(n)} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_{n'}, M) & \longrightarrow & K_{(n')} & \xrightarrow{\theta_{(n')}} & \mathbb{A}_{n'}^d & \longrightarrow & M_{(n')} & \longrightarrow & 0 \end{array} \tag{15}$$

in which each vertical map is the natural projection map. If, for each n, we write $\varrho_n = \varrho_n^d$ for the projection map $\mathbb{A}^d \to \mathbb{A}^d_{(n)} = \mathbb{A}^d_n$, then one has $\mathrm{im}(\theta_{(n)}) = \varrho_n(K)$ and so there is a natural exact sequence

$$0 \to \ker(\theta_{(n)}) \to K_{(n)} \to \varrho_n(K) \to 0. \tag{16}$$

In addition, since the left hand vertical map in the above diagram is surjective, by passing to the limit over $n > m$ in these diagrams we obtain the lower row in the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & K & \xrightarrow{\theta} & \mathbb{A}^d & \longrightarrow & M & \longrightarrow 0 \\
 & & \stackrel{\nu_1}{\swarrow} & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mu} & \\
\varprojlim_m K_{(m)} & & & & & & & & \\
 & & \stackrel{\nu_2}{\searrow} & & & & & & \\
 & 0 & \longrightarrow & \varprojlim_m \varrho_m(K) & \longrightarrow & \mathbb{A}^d & \longrightarrow & \varprojlim_m M_{(m)} & \longrightarrow 0,
\end{array} \tag{17}$$

in which ν_1, ν_2, ν and μ are the natural maps.

To prove claim (i), we are reduced (by Remark 2.8) to showing that if M is the limit of a pro-discrete system $(M_n, \pi_n)_n$, then the map μ is bijective. From the right hand square of the above diagram, it is clear that μ is surjective and to prove injectivity we proceed as follows. For each n one has an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\psi_n) & \longrightarrow & M_{(n)} & \xrightarrow{\psi_n} & M_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi'_n} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \pi_n} & & \\
0 & \longrightarrow & \ker(\psi_{n-1}) & \longrightarrow & M_{(n-1)} & \xrightarrow{\psi_{n-1}} & M_{n-1} & \longrightarrow & 0
\end{array}$$

in which ϕ_n, ψ_n and ψ_{n-1} are the natural projection maps and ϕ'_n is the restriction of ϕ_n. Here the surjectivity of ϕ'_n follows by applying the Snake Lemma to this diagram and noting that ψ_n maps $\ker(\phi_n)$ onto $\ker(\pi_n)$ (by the same argument as used after the diagram (8)). We may therefore pass to the limit over n of these diagrams to deduce that the map $\psi := \varprojlim_n \psi_n$ is a surjective homomorphism from $\varprojlim_n M_{(n)}$ to $\varprojlim_n M_n = M$, and hence that the composite $\psi \circ \mu$ is a surjective endomorphism of the R-module M. Then, since M is finitely generated, [43, Prop 1.2] implies that $\psi \circ \mu$, and hence also each of ψ and μ, is bijective. To complete the proof of claim (i) we finally note that, since ψ is bijective and each map ϕ'_n is surjective, every map ψ_n must itself be bijective, as required.

To prove claim (ii) we assume M is pro-discrete and hence, by claim (i), that the map μ is bijective. From the commutativity of (17), we can therefore deduce that ν is bijective, and hence that ν_2 is surjective, so that there exists a surjective map of $\mathbb{A}$-modules

$$\nu^{-1} \circ \nu_2 : \varprojlim_m K_{(m)} \twoheadrightarrow K.$$

To prove that the R-module K is finitely generated, and hence by (14) that M is finitely presented, it is therefore enough to show that the pro-discrete R-module $\varprojlim_m K_{(m)}$ is finitely generated. By Theorem 2.11(ii), we are therefore reduced to showing that the quantities $\mu_{R_n}(K_{(n)})$ are finite and bounded independently of n.

Next we note that the exact sequence (14) induces an identification

$$\ker(\theta_{(m)}) \cong \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_m, M) = \mathsf{Tor}^1_R(R_m, M). \tag{18}$$

In particular, since we are assuming $\mu_{R_m}(\mathsf{Tor}^1_R(R_m, M))$ to be finite and bounded independently of m, the exact sequence (16) therefore implies that the first assertion of claim (ii) is true if the quantities $\mu_{R_m}\big(\varrho_m(K)\big)$ are also both finite and bounded independently of m.

In addition, by applying the Forster-Swan Theorem (cf. [9, Th. 41.21]) over the $\mathbb{Z}_{\mathcal{V}}$-order R_m, one finds that the latter condition is satisfied if there exists a natural number c such that, for every m and every prime ideal $\mathfrak{p}$ of $\mathbb{Z}_{\mathcal{V}}$, one has

$$\mu_{R_{m,\mathfrak{p}}}\big(\varrho_m(K)_{\mathfrak{p}}\big) \le c. \tag{19}$$

If $\mathfrak{p} \neq p\mathbb{Z}_{\mathcal{V}}$, then $R_{m,\mathfrak{p}}$ is a finite product of principal ideal domains. Since $\varrho_m(K)_{\mathfrak{p}}$ is contained in $R_{m,\mathfrak{p}}[H]^d \cong (R_{m,\mathfrak{p}})^d[H]$ one therefore has $\mu_{R_{m,\mathfrak{p}}}\big(\varrho_m(K)_{\mathfrak{p}}\big) \le d\cdot|H|$.

To deal with the case $\mathfrak{p} = p\mathbb{Z}_{\mathcal{V}}$, we set $R'_n := R_{n,\mathfrak{p}}$, $N_n := \ker(\theta_{(n)})_{\mathfrak{p}}$ and $K_n := \varrho_n(K)_{\mathfrak{p}}$. Then the exact commutative diagram

$$\begin{array}{ccccccccc}
\mathsf{Tor}^1_{R'_m}(\mathbb{Z}_p, K_m) & \longrightarrow & \mathbb{Z}_p\otimes_{R'_m} N_m & \longrightarrow & \mathbb{Z}_p\otimes_{R'_m} K_{(m),\mathfrak{p}} & \longrightarrow & \mathbb{Z}_p\otimes_{R'_m} K_m & \longrightarrow & 0\\
 & & \big\downarrow & & \cong\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & N_0 & \longrightarrow & K_{(0),\mathfrak{p}} & \longrightarrow & K_0 & \longrightarrow & 0
\end{array} \tag{20}$$

(that is induced by comparing the $\mathfrak{p}$-localisations of the sequences (16) for $n = m$ and $n = 0$) gives rise to an exact sequence

$$\mathsf{Tor}^1_{R'_m}(\mathbb{Z}_p, K_m) \to \mathbb{Z}_p\otimes_{R'_m} N_m \to N_0 \to \mathbb{Z}_p\otimes_{R'_m} K_m \to K_0 \to 0. \tag{21}$$

In particular, since the kernel of the projection $R'_m \to \mathbb{Z}_p$ is contained in the Jacobson radical of R'_m, Nakayama's Lemma implies that, in this case, one has

$$\mu_{R_{m,\mathfrak{p}}}\big(\varrho_m(K)_{\mathfrak{p}}\big) = \mu_{\mathbb{Z}_p}\big(\mathbb{Z}_p\otimes_{R'_m} K_m\big) \le \mu_{\mathbb{Z}_p}(N_0) + \mu_{\mathbb{Z}_p}(K_0).$$

The required inequality (19) is therefore satisfied if one takes c to be the maximum of $d\cdot|H|$ and the sum $\mu_{\mathbb{Z}_p}(N_0) + \mu_{\mathbb{Z}_p}(K_0)$. This proves the backwards implication of the first assertion in claim (ii).

To verify the corresponding forwards implication, we assume M is finitely presented so that the quantity $t := \mu_R(K)$ is finite. Then the diagram (15) combines with (18) to identify $\mathsf{Tor}^1_R(R_n, M) \cong \ker(\theta_{(n)})$ with a submodule of the $\mathbb{A}_n$-module $K_{(n)}$. For each prime $\ell \neq p$ one therefore has $\mu_{R_{n,\ell}}(\mathsf{Tor}^1_R(R_n, M)_\ell) \le t$ and so, just as above, it is enough to show the quantity $\mu_{R_{n,p}}(\ker(\theta_{(n)})_p)$ is bounded independently of n. In view of the sequence (21), we are thus reduced (by Nakayama's Lemma) to showing that $\mu_{\mathbb{Z}_p}(\mathsf{Tor}^1_{R'_n}(\mathbb{Z}_p, K_n))$ is bounded independently of n. To verify this we note that there are isomorphisms

$$\mathsf{Tor}^1_{R'_n}(\mathbb{Z}_p, K_n) \cong \widehat{H}^{-2}(\Gamma_n, K_n) \cong \widehat{H}^0(\Gamma_n, K_n),$$

where the first isomorphism is clear and the second follows from the periodicity of Tate cohomology for cyclic groups. It is therefore enough to note that $\widehat{H}^0(\Gamma_n, K_n)$ is a quotient of the $\mathbb{Z}_p$-submodule $K_n^{\Gamma_n}$ of $(R_{n,p}^d)^{\Gamma_n} \cong \mathbb{Z}_p^d$ and hence that $\mu_{\mathbb{Z}_p}\big(\widehat{H}^0(\Gamma_n, K_n)\big) \le d$. This completes the proof of the first assertion in claim (ii).

To prove the final assertion of claim (ii), we note that, for each n, the short exact sequence in Proposition 2.1(iii) induces a long exact sequence

$$\begin{aligned} 0 \to \mathsf{Tor}_R^2(R_n, M) \to \mathsf{Tor}_R^1((\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} R_n, M) \\ \to M^{\varpi_n = 0} \to \mathsf{Tor}_R^1(R_n, M) \to \big(\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}\big) \otimes_{\mathbb{Z}_\mathcal{V}} M_{(n)}. \end{aligned} \quad (22)$$

Here we also use the fact that the short exact sequence in Proposition 2.1(i) implies that $\mathsf{Tor}_R^2(R_{[n]}, M)$ vanishes and also induces an identification of $\mathsf{Tor}_R^1(R_{[n]}, M)$ with $M^{\varpi_n=0}$.

In particular, if $M_n \cong M_{(n)}$ has finite exponent, then the group $\big(\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}\big) \otimes_{\mathbb{Z}_\mathcal{V}} M_{(n)}$ vanishes and so $\mathsf{Tor}_R^1(R_n, M)$ is isomorphic to a quotient of $M^{\varpi_n=0}$. From here, the final assertion of claim (ii) is clear.

To prove claim (iii) we note that, under the given hypothesis, the canonical homomorphism $\mu : M \to \varprojlim_n M_{(n)}$ lies in a commutative diagram of $\mathbb{\Lambda}$-modules of the form

$$\begin{array}{ccc} M & \xrightarrow{\subseteq} & \mathbb{\Lambda}^a \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow \\ \varprojlim_n M_{(n)} & \longrightarrow & \varprojlim_n \mathbb{\Lambda}_n^a. \end{array}$$

In particular, since the right hand vertical arrow is bijective, the map μ is injective. By constructing a commutative diagram of the form (17), we can then deduce firstly that μ is bijective, and hence that M is pro-discrete (as claimed), and also that any module K that occurs in an exact sequence of the form (14) identifies with $\varprojlim_n \varrho_n(K)$.

In particular, to deduce the second assertion of claim (iii) from claim (ii), it is enough to note that the tautological exact sequence

$$0 \to M \to \mathbb{\Lambda}^a \to \mathbb{\Lambda}^a/M \to 0$$

induces a canonical isomorphism of $\mathbb{\Lambda}$-modules

$$\mathsf{Tor}_R^1(R_n, M) \cong \mathsf{Tor}_R^2(R_n, \mathbb{\Lambda}^a/M)$$

and hence implies $\mu_R\big(\mathsf{Tor}_R^1(R_n, M)\big) = \mu_R\big(\mathsf{Tor}_R^2(R_n, \mathbb{\Lambda}^a/M)\big)$.

In addition, to prove the final assertion of claim (iii) it is enough to show the stated conditions imply that, for each $n > n'$, the map $\kappa_{n'}^n : R_{n'} \otimes_{R_n} \varrho_n(K) \to \varrho_{n'}(K)$ that is induced by the natural surjective map $\varrho_{n,n'} : \varrho_n(K) \to \varrho_{n'}(K)$ is injective (and hence bijective). Indeed, if this is true, then K is the limit of the pro-discrete family $(\varrho_n(K), \varrho_{n,n-1})_n$. Thus, since the argument in claim (ii) shows that the quantities $\mu_{R_n}(\varrho_n(K))$ are finite and bounded independently of n, the result of Theorem 2.11(ii) implies that K is finitely generated, and hence that M is finitely presented, as required.

To prove injectivity of $\kappa_{n'}^n$ we set $\Delta_{n'}^n := \Gamma^{p^{n'}}/\Gamma^{p^n}$ and note that the functor $R_{n'} \otimes_{R_n} -$ identifies with taking $\Delta_{n'}^n$-coinvariants. In particular, the diagram (15) gives rise to an exact commutative diagram

$$\begin{array}{ccccccccc} \mathsf{Tor}^1_{R_n}(R'_n, M_{(n)}) & \xrightarrow{\tilde\kappa^n_{n'}} & \big(\varrho_n(K)\big)_{\Delta^n_{n'}} & \xrightarrow{(\theta_{(n)})_{\Delta^n_{n'}}} & \big(\mathbb{A}^d_n\big)_{\Delta^n_{n'}} & \longrightarrow & \big(M_{(n)}\big)_{\Delta^n_{n'}} & \longrightarrow & 0 \\ & & \big\downarrow \kappa^n_{n'} & & \big\downarrow\cong & & \big\downarrow & & \\ 0 & \longrightarrow & \varrho_{n'}(K) & \xrightarrow{\theta_{(n')}} & \mathbb{A}^d_{n'} & \longrightarrow & M_{(n')} & \longrightarrow & 0, \end{array}$$

which implies that $\ker(\kappa^n_{n'}) = \ker((\theta_{(n)})_{\Delta^n_{n'}}) = \mathrm{im}(\tilde\kappa^n_{n'})$ is isomorphic to a quotient of the group $\mathsf{Tor}^1_{R_n}(R'_n, M_{(n)}) \cong H^{-2}(\Delta^n_{n'}, M_{(n)})$. In particular, since the exponent of the latter group divides $|\Delta^n_{n'}| = p^{n-n'}$, the same is also true for the group $\ker(\kappa^n_{n'})$.

On the other hand, the short exact sequence (16) combines with the identifications $\ker(\theta_{(n)}) \cong \mathsf{Tor}^1_R(R_n, M) \cong \mathsf{Tor}^2_R(R_n, \mathbb{A}^a/M)$ to give an exact commutative diagram

$$\begin{array}{ccccccccc} & & \big(\mathsf{Tor}^2_R(R_n, \mathbb{A}^a/M)\big)_{\Delta^n_{n'}} & \longrightarrow & \big(K_{(n)}\big)_{\Delta^n_{n'}} & \longrightarrow & \big(\varrho_n(K)\big)_{\Delta^n_{n'}} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow\cong & & \big\downarrow\kappa^n_{n'} & & \\ 0 & \longrightarrow & \mathsf{Tor}^2_R(R_{n'}, \mathbb{A}^a/M) & \longrightarrow & K_{(n')} & \longrightarrow & \varrho_{n'}(K) & \longrightarrow & 0, \end{array}$$

which implies $\ker(\kappa^n_{n'})$ is isomorphic to a quotient of $\mathsf{Tor}^2_R(R_{n'}, \mathbb{A}^a/M)$. To prove that $\ker(\kappa^n_{n'})$ vanishes, it thus suffices to show that the group $\mathsf{Tor}^2_R(R_{n'}, \mathbb{A}^a/M)$ is p-divisible.

It is therefore enough to note that the group $\mathsf{Tor}^1_R((\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} R_{n'}, \mathbb{A}^a/M)$ is uniquely p-divisible and hence that, if $\big(\mathbb{A}^a/M\big)^{\varpi_n=0}[p^\infty]$ has finite exponent, then the exact sequence (22) (with M replaced by $\mathbb{A}^a/M$) implies that $\mathsf{Tor}^2_R(R_{n'}, \mathbb{A}^a/M)$ is p-divisible.

This completes the proof of Proposition 2.13. □

Remark 2.14. Proposition 2.13(i) implies that there exist finitely presented R-modules that are not pro-discrete. For example, if M is the cyclic R-module R/J, then, for each n, the module $M_{(n)}$ identifies with R/I and so the limit $\varprojlim_n M_{(n)}$ also identifies with R/I. However, the natural map $R/J = M \to \varprojlim_n M_{(n)} = R/I$ is not injective and so M is not pro-discrete.

Remark 2.15. Let M be an $R_n[H]$-module (for some n). Then the natural map $M \to M_{(n)}$ is bijective and so M is equal to the pro-discrete $R[H]$-module $\varprojlim_n M_{(n)}$. If M is finite, then it validates the explicit condition in the final assertion of Proposition 2.13(ii) and so is a finitely presented $R[H]$-module. However, if M is not finite and $\mathcal{V} = \varnothing$, then it is not finitely-presented since, as a module over the subalgebra $\mathbb{Z}[[\Gamma^{p^n}]]$ (over which $R[H]$ is finitely generated), M has a direct summand isomorphic to $\mathbb{Z}$ and this is not finitely presented (by Remark 2.3(i)).

Remark 2.16. The Forster-Swan Theorem plays a key role in the proof of Proposition 2.13. By using this result, one can also easily derive the following useful fact: if M is any $R_n[H]$-module that is locally-free of rank d, then one has $\mu_{R_n[H]}(M) \le d+1$.

2.4. **The finitely generated spectrum of $\mathbb{Z}[[\mathbb{Z}_p]]$.** In this subsection we record consequences of Proposition 2.13 concerning the structure of ideals of R. In this way we shall, in particular, obtain a full, and explicit, description of the finitely generated prime spectrum of R (and hence, as a special case, of $\mathbb{Z}[[\mathbb{Z}_p]]$).

We start, however, with a useful general result.

Proposition 2.17. *The following claims are valid for every ideal M of R.*

(i) *If $M \not\subseteq I$, then $\mathbb{Z}_\mathcal{V}/\varrho_0(M)$ is finite and the following claims are valid.*
  (a) *The ideal $I+M$ is finitely presented and such that*
$$I+M \text{ is principal} \iff |\mathbb{Z}_\mathcal{V}/\varrho_0(M)| \text{ is prime to } p.$$
  (b) *The ideal $I \cap M$ is not finitely generated.*

(ii) *If $M \subseteq I$, then the following claims are valid.*
  (a) *If $J \subseteq M$, then M is finitely presented if and only if J has finite index in M and, if this is the case, then M is either principal or has free dimension one.*
  (b) *If $J \not\subseteq M$ and $J \cap M$ is finitely presented, then M is finitely presented if and only if $J \cap M$ has finite index in M.*

Proof. To prove claim (i) we set $M' := I + M$.

The first assertion of claim (i) is clear since, if $M \not\subset I$, then $\varrho_0(M)$ is non-zero. The quotient module $\mathbb{Z}_\mathcal{V}/\varrho_0(M) \cong R/M'$ is therefore finite and we write t for its order.

Then $M' = I + tR$ and so Proposition 2.1(iii) implies that M' is equal to $R(\omega_0, t)$ and is thus finitely generated. Since R/M' is finite, the last assertion of Proposition 2.13(iii)(b) then implies M' is finitely presented, as required to prove the first assertion of claim (i)(a).

We next note that if M' is R-projective, then $M'_{(n)}$ is R_n-projective. Since $\mathsf{Tor}^1_R(R_n, \mathbb{Z}_\mathcal{V}/(t))$ is a torsion-group, the tautological exact sequence

$$0 \to M' \xrightarrow{\subseteq} R \xrightarrow{1 \mapsto 1_t} \mathbb{Z}_\mathcal{V}/(t) \to 0, \tag{23}$$

in which 1_t denotes the residue class of 1, therefore induces, for each n, an exact sequence

$$0 \to M'_{(n)} \to R_n \to \mathbb{Z}/(t) \to 0.$$

This sequence implies $\mathbb{Z}_\mathcal{V}/(t)$ is a cohomologically-trivial Γ_n-module and hence that t must be prime-to-p. To complete the proof of claim (i)(a), it is therefore enough to show that, if $t > 1$ is prime-to-p, then M' is principal. To show this, for each n we write γ_n for the image of γ in Γ_n, e_n for the idempotent $p^{-n}\sum_{g\in\Gamma_n} g$ of $\mathbb{Q}[\Gamma_n]$ and $x_{t,n}$ for the element $e_n + y_n(1-e_n)$ of $\mathbb{Q}[\Gamma_n]$, where y_n is the unique element of $\mathbb{Q}[\Gamma_n](1-e_n)$ such that

$$(\gamma_n^t - 1)y_n = t(\gamma_n - 1).$$

Then, for each $n > 2$, there exists an exact commutative diagram of R_n-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_n & \xrightarrow{1\mapsto t} & R_n \cdot x_{t,n} & \xrightarrow{x_{t,n}\mapsto 1_t} & \mathbb{Z}_\mathcal{V}/(t) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle x_{t,n}\mapsto x_{t,n-1}} & & \Big\| & & \\
0 & \longrightarrow & R_{n-1} & \xrightarrow{1\mapsto t} & R_{n-1} \cdot x_{t,n-1} & \xrightarrow{x_{t,n-1}\mapsto 1_t} & \mathbb{Z}_\mathcal{V}/(t) & \longrightarrow & 0
\end{array}$$

in which π_n is the natural projection map. Here the fact that $t \in R_n \cdot x_{t,n}$ and exactness of the rows follow from applying the functor $- \otimes_\mathbb{Z} \mathbb{Z}_\mathcal{V}$ to the corresponding diagram with $\mathcal{V} = \varnothing$, which itself exists as a consequence of the discussion about Swan modules in [9,

§53A]. In particular, each map π_n is surjective and each R_n-module $R_n \cdot x_{t,n}$ is free of rank one with basis $x_{t,n}$. Thus, by passing to the limit over n in these diagrams, we obtain an exact sequence of R-modules

$$0 \to R \to R \xrightarrow{1 \mapsto 1_t} \mathbb{Z}_{\mathcal{V}}/(t) \to 0.$$

By comparing this sequence to (23) we deduce that the ideal M' is isomorphic to R, and so is principal, as claimed. This completes the proof of claim (i)(a).

To prove claim (i)(b) we use the natural short exact sequence of R-modules

$$0 \to I \cap M \to M \oplus I \to M' \to 0.$$

In particular, since M' is finitely generated (by claim (i)(a)), and I is not finitely generated (by Proposition 2.1(ii)), this sequence implies $I \cap M$ cannot be finitely generated, as required.

Turning to claim (ii)(a) we note that, since J is isomorphic to R, the short exact sequence

$$0 \to J \to M \to M/J \to 0$$

combines with the general result of [15, Th. 2.1.2] to imply that M is finitely presented if and only if M/J is finitely presented. In addition, since $M \subseteq I$, the exact sequence in Proposition 2.1(ii) (with $n = 0$) implies that the action of R on M/J factors through the projection $\varrho_0 : R \to \mathbb{Z}_{\mathcal{V}}$. In particular, if M is a finitely generated R-module, then M/J is isomorphic, as an R-module, to a finitely generated submodule of $\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}$ and so is isomorphic to a direct sum $\mathbb{Z}_{\mathcal{V}}^s \oplus \mathbb{Z}_{\mathcal{V}}/(t)$ (upon which Γ acts trivially) for suitable non-negative integers s and t. It follows that M/J is finite if and only if $s = 0$. From [15, Th. 2.1.2] we also know that M/J is finitely presented if and only if each occurring factor $\mathbb{Z}_{\mathcal{V}}$ and $\mathbb{Z}_{\mathcal{V}}/(t)$ is finitely presented.

The first assertion of claim (ii)(a) is therefore true since Proposition 2.1(i), respectively (iv), implies that $\mathbb{Z}_{\mathcal{V}}$ is not finitely presented, respectively $\mathbb{Z}_{\mathcal{V}}/(t)$ is finitely presented. In addition, since any torsion element of $\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}$ has order prime-to-p, if the index of J in M is finite, then M/J is isomorphic to $\mathbb{Z}_{\mathcal{V}}/(t)$ for a natural number t prime-to-p. The argument of claim (i) thus gives an exact sequence of R-modules of the form $0 \to R \to R \to M/J \to 0$ and, since J is isomorphic to R, this sequence implies that M has free dimension at most one, as required.

The result of claim (ii)(b) follows upon applying [15, Th. 2.1.2] to the exact sequences

$$0 \to J \cap M \to M \to (J + M)/J \to 0 \quad \text{and} \quad 0 \to M \to J + M \to (J + M)/J \to 0,$$

and then using the result of claim (ii)(a) with M replaced by $J + M$. □

We now turn to consider prime ideals of R. We note, in particular, that claims (iii) and (iv) of the following result give a complete classification of the finitely generated prime spectrum of R.

Theorem 2.18. *If $\mathfrak{p}$ is a prime ideal of R, then the following claims are valid.*

(i) *If the canonical map $\mathfrak{p} \to \varprojlim_n \mathfrak{p}_{(n)}$ is bijective, then $\mathfrak{p}$ is the contraction of an ideal in Λ. In particular, this is true if $\mathfrak{p}$ is finitely generated. In addition, unless $\mathfrak{p}$ contains ϖ_n for some n, the natural map $\mathfrak{p}_{(m)} \to \varrho_m(\mathfrak{p})$ is bijective for every natural number m.*

(ii) *$\mathfrak{p}$ is finitely presented if and only if it is finitely generated.*

(iii) *If* $\mathfrak{p}$ *contains* ϖ_n *for some* n*, then the following conditions are equivalent:*
 (a) $\mathfrak{p}$ *is finitely generated;*
 (b) $\mathfrak{p}$ *is maximal;*
 (c) $\mathfrak{p}$ *has finite index in* R*.*

(iv) *If* $\mathfrak{p}$ *does not contain* ϖ_n *for any* n*, then it is finitely generated if and only if it is the contraction of a prime ideal of* $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$ *that is generated by an irreducible distinguished polynomial* f *that is not equal to* $(1+T)^a - 1$ *for any* a *in* $p\mathbb{Z}_p$*. If this is the case, then one has* $\Lambda\mathfrak{p} = \Lambda f$*.*

Proof. We write $\overline{\mathfrak{p}}$ for the ideal $\varprojlim_n \varrho_n(\mathfrak{p})$ of R. We note that $\overline{\mathfrak{p}}$ is the contraction of an ideal of Λ and that $\mathfrak{p} \subseteq \overline{\mathfrak{p}}$, with equality if and only if $\mathfrak{p}$ is the contraction of an ideal of Λ. For each n we also set $\overline{\mathfrak{p}}_n := \varrho_n(\mathfrak{p})$.

To verify the first assertion of claim (i) we assume the canonical map $\mathfrak{p} \to \varprojlim_n \mathfrak{p}_{(n)}$ is bijective. Then, setting

$$Q := R/\mathfrak{p} \quad \text{and} \quad Q_n[1] := \mathsf{Tor}^1_R(Q, R_n) \text{ for each } n,$$

we may pass to the limit over n of the natural exact sequences

$$0 \to Q_n[1] \to \mathfrak{p}_{(n)} \to \overline{\mathfrak{p}}_n \to 0 \tag{24}$$

(cf. (16)) to obtain an exact sequence

$$0 \to \mathfrak{p} \to \overline{\mathfrak{p}} \to \varprojlim\nolimits^1_n Q_n[1] \to 0.$$

In addition, since each R_n-module $Q_n[1]$ is annihilated by $\overline{\mathfrak{p}}_n$, the action of R on $\varprojlim^1_n Q_n[1]$ factors through $R/\overline{\mathfrak{p}}$. From the last displayed exact sequence we may therefore deduce that $\overline{\mathfrak{p}}^2 \subseteq \mathfrak{p}$, and hence, since $\mathfrak{p}$ is prime, that $\mathfrak{p} = \overline{\mathfrak{p}}$, as required.

To prove the final assertion of claim (i) we recall that if $\mathfrak{p}$ is finitely generated, then it is pro-discrete by Proposition 2.13(iii) and so Proposition 2.13(i) implies that the canonical map $\mathfrak{p} \to \varprojlim_n \mathfrak{p}_{(n)}$ is bijective. Furthermore, if m is any natural number then the sequence (24) implies that $\mathsf{Tor}^1_R(Q, R_m)$ is countable. Since the above argument establishes that the derived limit $\varprojlim^1_m Q_m[1]$ vanishes, one deduces that the inverse system $(Q_m[1], \tau_{m,m'})_{m,m'\in\mathbb{N}}$ is Mittag-Leffler by [16, p. 242, Prop.], where the $\tau_{m,m'} : Q_m[1] \to Q_{m'}[1]$ are the natural maps induced by the sequences (24).

In particular, since the limit $\varprojlim_m Q_m[1]$ also vanishes, there exists a natural number $n > m$ for which $\tau_{n,m}$ is the zero map. Hence, if we write $\Gamma_n^{(a)}$ for the unique subgroup of Γ_n of order p^a for each integer a with $0 \le a \le n$, then a comparison of the $\Gamma_n^{(n-m)}$-coinvariants of the exact sequence (24) with the corresponding sequence with n replaced by m shows that $Q_m[1]$ is a finite group of p-power order. Then, since $\mathfrak{p}$ is assumed not to contain ϖ_m, the exact sequence (22) combines with the fact that $\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}$ is uniquely p-divisible to imply that $Q_m[1]$ vanishes, and hence that the map $\mathfrak{p}_{(m)} \to \overline{\mathfrak{p}}_m$ is bijective, as claimed.

Claim (ii) follows directly from Proposition 2.13(iii) and the fact that, since $R/\mathfrak{p}$ is an integral domain, the submodule $(R/\mathfrak{p})[p^\infty]$ is either trivial or of exponent p.

We note next that if $\mathfrak{p}$ contains ϖ_n, and hence J_n, then Proposition 2.1(iv) implies $I_n^2 \subseteq J_n \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, one therefore has $I_n \subseteq \mathfrak{p}$ and so $\overline{\mathfrak{p}}_n$ is a prime ideal of R_n that is maximal if and only if $\mathfrak{p}$ is a maximal ideal of R. In particular, since any maximal ideal of R_n has finite index, claim (iii) is true if we can show that $\mathfrak{p}$ is finitely generated if and only if $\overline{\mathfrak{p}}_n$ has finite index in R_n.

Now, if $\overline{\mathfrak{p}}_n$ has finite index in R_n, then it contains a prime ℓ. By claim (i), one must have $\ell = p$ and so $\mathfrak{p}$ contains $I_n + pR$. In particular, since the latter ideal is finitely generated (by Proposition 2.1(iv)), and the quotient of $\overline{\mathfrak{p}}_n$ by pR_n is finite, the exact sequence

$$0 \to I_n + pR \to \mathfrak{p} \to \overline{\mathfrak{p}}_n/(pR_n) \to 0$$

implies $\mathfrak{p}$ is finitely generated, as required.

To prove the converse, we argue by contradiction and thus assume both that $\overline{\mathfrak{p}}_n$ has infinite index in R_n and $\mathfrak{p}$ is finitely generated. We then fix an element x_n of $\mathfrak{p}$ with the property that the R_n-module $(\overline{x_n})$ generated by $\varrho_n(x_n)$ has finite index in $\overline{\mathfrak{p}}_n$. By applying the Snake Lemma to the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_n & \longrightarrow & R & \xrightarrow{\varrho_n} & R_n & \longrightarrow & 0 \\
 & & \downarrow{\cdot x_n} & & \downarrow{\cdot x_n} & & \downarrow{\nu} & & \\
0 & \longrightarrow & I_n & \longrightarrow & \mathfrak{p} & \xrightarrow{\varrho_n} & \overline{\mathfrak{p}}_n & \longrightarrow & 0,
\end{array}$$

where ν denotes multiplication by $\varrho_n(x_n)$, we obtain an exact sequence of R-modules

$$\ker(\nu) \to I_n/(I_n x_n) \to \mathfrak{p}/(Rx_n) \to \overline{\mathfrak{p}}_n/(\overline{x_n}) \to 0.$$

In particular, since $\overline{\mathfrak{p}}_n/(\overline{x_n})$ is finite (and hence finitely presented, by Remark 2.15), and both $\ker(\nu)$ and $\mathfrak{p}/(Rx_n)$ are finitely generated R-modules (the former since R_n is Noetherian and the latter by hypothesis), this sequence combines with the general result of [15, Th. 2.1.2] to imply that the R-module $I_n/(I_n x_n)$ is finitely generated. However, this cannot be true since the exact sequence in Proposition 2.1(ii) implies the existence of a surjective homomorphism of R-modules from $I_n/(I_n x_n)$ to $\big(\mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}\big) \otimes_{\mathbb{Z}_{\mathcal{V}}} (R_n/(\overline{x_n}))$ and the latter module is not finitely generated over R (since, by assumption, the $\mathbb{Z}_{\mathcal{V}}$-module $R_n/(\overline{x_n})$ has a non-zero torsion-free quotient). This contradiction completes the proof of claim (iii).

To prove claim (iv) we fix in the rest of the argument a prime ideal $\mathfrak{p}$ that does not contain ϖ_n for any n.

We first assume $\mathfrak{p}$ is finitely generated so that by claim (i) the natural map $\mathfrak{p} \to \varprojlim_n \mathfrak{p}_{(n)}$ is bijective. As an initial step in the proof of (iv) we claim that each $\mathfrak{p}_{(n)}$ is an invertible R_n-module. Note that this module is $\mathbb{Z}_{\mathcal{V}}$-torsion-free since (i) also implies that the natural map $\mathfrak{p}_{(n)} \to \overline{\mathfrak{p}}_n$ is bijective. It therefore suffices (by Swan's Theorem) to show that $\mathfrak{p}_{(n)}$ is a cohomologically-trivial Γ_n-module.

To do this we fix a natural number m with $m \leq n$ and note first that, since $\mathfrak{p}$ is pro-discrete, the natural map

$$\mathbb{Z}_{\mathcal{V}} \otimes_{\mathbb{Z}_{\mathcal{V}}[\Gamma_n^{(m)}]} \mathfrak{p}_{(n)} = R_{n-m} \otimes_{R_n} \mathfrak{p}_{(n)} \to \mathfrak{p}_{(n-m)}$$

is bijective. It follows that the group $\widehat{H}^{-1}(\Gamma_n^{(m)}, \mathfrak{p}_{(n)})$ is isomorphic to a finite subgroup of $\mathfrak{p}_{(n-m)} = \overline{\mathfrak{p}}_{n-m}$ and so must vanish.

On the other hand, the structure theorem for finitely generated Λ-modules implies the existence of an exact sequence of Λ-modules of the form $0 \to \Lambda\mathfrak{p} \to \Lambda \to N \to 0$ in which

N is a finite p-group (but the second map need not be inclusion). By applying the functor $R_{n,p}\otimes_\Lambda -$ to this sequence one obtains an exact sequence of $R_{n,p}$-modules

$$0\to \mathrm{Tor}^1_\Lambda(R_{n,p},N)\to (\Lambda\mathfrak{p})_{(n)}\to R_{n,p}\to R_{n,p}\otimes_\Lambda N\to 0.$$

This sequence implies that the (finitely generated) $\mathbb{Z}_p[\Gamma_n^{(m)}]$-module $\mathfrak{p}_{(n),p}=(\Lambda\mathfrak{p})_{(n)}$ spans the free $\mathbb{Q}_p[\Gamma_n^{(m)}]$-module $\mathbb{Q}_p[\Gamma_n]=\mathbb{Q}_p\otimes_{\mathbb{Z}_p}R_{n,p}$ and hence has vanishing Herbrand quotient with respect to the action of the (cyclic) group $\Gamma_n^{(m)}$. Hence, since $\widehat{H}^{-1}(\Gamma_n^{(m)},\mathfrak{p}_{(n)})$ vanishes, the group $\widehat{H}^0(\Gamma_n^{(m)},\mathfrak{p}_{(n)})$ also vanishes, and so (since this is true for all $m\le n$) the Γ_n-module $\mathfrak{p}_{(n)}$ is cohomologically-trivial, as claimed.

The above discussion implies that the ideal $\overline{\mathfrak{p}}_n=\varrho_n(\mathfrak{p})$ of R_n is invertible and hence has finite index. By passing to the limit over n of the tautological exact sequences

$$0\to\varrho_n(\mathfrak{p})\to R_n\to Q(n)\to 0, \tag{25}$$

one therefore obtains an identification

$$R/\mathfrak{p}\cong\varprojlim_n Q(n)\cong\prod_\ell\bigl(\varprojlim_n Q(n,\ell)\bigr), \tag{26}$$

where ℓ runs over all primes and $Q(n,\ell)$ is the Sylow ℓ-subgroup of $Q(n)$. Now, since $R/\mathfrak{p}$ is an integral domain, only one component in this direct product can be non-zero. Since the natural maps $Q(n,\ell)\to Q(m,\ell)$ are surjective for all $n>m$, it follows that there exists a prime ℓ such that every group $Q(n)$ has ℓ-power order.

Suppose, firstly, that $\ell\neq p$. The sequence (25) implies that, for each n, the module $Q(n)$ is a cohomologically trivial R_n-module and, therefore, that both $(Q(n),\tau_n)$ and $(\ell\cdot Q(n),\ell\cdot\tau_n)$ are pro-discrete systems where for each n, we write $\tau_n : Q(n)\to Q(n-1)$ for the natural maps induced by (25). This fact can then be used to deduce that the natural map $Q(n)[\ell]\to Q(n-1)[\ell]$ induced by τ_n is surjective. In particular, $R/\mathfrak{p}$ has an element of order ℓ and hence $\ell\in\mathfrak{p}$. This contradicts the result of claim (i).

It follows that $\ell=p$ and to deal with this case we write $\varrho_{n,p}$ for the natural projection $\Lambda\to\mathbb{Z}_p[\Gamma_n]$. Then, for every n, one has

$$R_n/\varrho_n(\mathfrak{p})=\mathbb{Z}_p\otimes_{\mathbb{Z}_{\mathcal{V}}}\bigl(R_n/\varrho_n(\mathfrak{p})\bigr)=\mathbb{Z}_p[\Gamma_n]/(\mathbb{Z}_p\cdot\varrho_n(\mathfrak{p}))=\mathbb{Z}_p[\Gamma_n]/\varrho_{n,p}(\Lambda\mathfrak{p})$$

and hence also $\varrho_n(\mathfrak{p})=R_n\cap\varrho_{n,p}(\Lambda\mathfrak{p})$. By passing to the limit over n, these identifications imply that

$$R/\mathfrak{p}=R/\overline{\mathfrak{p}}=\varprojlim_n\bigl(R_n/\varrho_n(\mathfrak{p})\bigr)=\varprojlim_n\bigl(\mathbb{Z}_p[\Gamma_n]/\varrho_{n,p}(\Lambda\mathfrak{p})\bigr)=\Lambda/\Lambda\mathfrak{p}$$

so that $\Lambda\mathfrak{p}$ is a prime ideal of Λ, and also $\mathfrak{p}=R\cap\Lambda\mathfrak{p}$.

In addition, under the present hypotheses, $\Lambda\mathfrak{p}$ cannot be the maximal ideal of Λ since $\omega_0\notin\mathfrak{p}$. We can therefore assume that $\Lambda\mathfrak{p}=\Lambda f$ for an irreducible distinguished polynomial f. The requirement that $\varrho_{n,p}(\Lambda\mathfrak{p})$ has finite index in $\mathbb{Z}_p[\Gamma_n]$ for every n is then equivalent to the condition that f is not equal to $(1+T)^a-1$ for any element a of $p\mathbb{Z}_p$.

At this stage we know that if $\mathfrak{p}$ is finitely generated, then it has the explicit properties stated in claim (iv). To complete the proof, it is therefore enough to show that if f is any irreducible distinguished polynomial as above, then the contracted prime ideal $\mathfrak{p} := R \cap \Lambda f$ of R is finitely generated and such that $\Lambda\mathfrak{p} = \Lambda f$.

In this case, the given condition on f implies that, for every n, the index of $\varrho_{n,p}(\Lambda f)$ in $\mathbb{Z}_p[\Gamma_n]$ is finite and so $R_n + \varrho_{n,p}(\Lambda f) = \mathbb{Z}_p[\Gamma_n]$. Setting $\mathfrak{q}_n := R_n \cap \varrho_{n,p}(\Lambda f)$, it follows that $\mathbb{Z}_p \cdot \mathfrak{q}_n = \varrho_{n,p}(\Lambda f)$ (so that $\Lambda\mathfrak{p} = \Lambda f$) and also that there exists an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{q}_n & \xrightarrow{x\to(x,x)} & R_n \oplus \varrho_{n,p}(\Lambda f) & \xrightarrow{(x,y)\mapsto x-y} & \mathbb{Z}_p[\Gamma_n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{q}_{n-1} & \xrightarrow{x\to(x,x)} & R_{n-1} \oplus \varrho_{n-1,p}(\Lambda f) & \xrightarrow{(x,y)\mapsto x-y} & \mathbb{Z}_p[\Gamma_{n-1}] & \longrightarrow & 0 \end{array}$$

in which every vertical map is the natural projection.

The $\mathbb{Z}_p[\Gamma_n]$-module $\varrho_{n,p}(\Lambda f)$ is free of rank one. This fact combines with the exactness of the upper row in the above diagram to imply that the lattice $\mathfrak{q}_n$ is a cohomologically-trivial Γ_n-module, and hence an invertible (that is, locally-free of rank one) R_n-module, and also means that the Snake Lemma can be applied to the diagram to show that the natural map $R_{n-1} \otimes_{R_n} \mathfrak{q}_n \to \mathfrak{q}_{n-1}$ is bijective. The latter isomorphisms imply that the ideal $\mathfrak{p} = \varprojlim_n \mathfrak{q}_n$ of R is pro-discrete.

To deduce $\mathfrak{p}$ is finitely generated, it is now enough to apply Theorem 2.11(ii) after noting that, for every n, the invertible R_n-module $\mathfrak{q}_n$ is such that $\mu_{R_n}(\mathfrak{q}_n) \leq 2$ (cf. Remark 2.16).

This completes the proof of Theorem 2.18. □

Remark 2.19. Aside from its upcoming role in the proof of Theorem 1.1, having an explicit characterisation of finitely generated prime ideals can be useful in other ways. For example, Anderson [2] has shown that, in any commutative ring, if every prime ideal that is minimal over a given ideal M is finitely generated, then there can be only finitely many prime ideals that are minimal over M.

2.5. **Strong 2-coherence and the proof of Theorem 1.1.** In this subsection we consider the various assertions of Theorem 1.1 relative to the algebra $R = \mathbb{Z}_{\mathcal{V}}[[\mathbb{Z}_p]]$ for a finite set of rational primes $\mathcal{V}$ that does not contain p. The proof of Theorem 1.1 itself then follows upon specialisation of the results below to the case that $\mathcal{V} = \varnothing$ so $R = \mathbb{Z}[[\mathbb{Z}_p]]$.

We start by proving a result concerning the notion of 'strong n-coherence', the definition of which can be found in [12, Def. 2.2 and Rem. 2.3] (and is also reviewed in the proof given below).

Proposition 2.20. *For any finite abelian group H, the following claims are valid.*

(i) *The ring $R[H]$ is strong 2-coherent.*
(ii) *An $R[H]$-module is finitely ∞-presented if and only if it is finitely 2-presented.*
(iii) *If N is any finitely generated pro-discrete $R[H]$-module, then it is finitely ∞-presented if the quantity*

$$\max\{\mu_{R_n}\big(\mathsf{Tor}^1_R(R_n, N)\big),\, \mu_{R_n}\big(\mathsf{Tor}^2_R(R_n, N)\big)\}$$

is finite and bounded independently of n.

Proof. By definition, a ring is strong 2-coherent if every finitely 2-presented module is finitely 3-presented. To prove claim (i), it is therefore enough to show that, in any exact sequence of $R[H]$-modules of the form

$$R[H]^{d_2} \xrightarrow{\theta_2} R[H]^{d_1} \xrightarrow{\theta_1} R[H]^{d_0} \to M \to 0 \tag{27}$$

(where d_0, d_1 and d_2 are natural numbers), the module $K_1 := \ker(\theta_1) = \mathrm{im}(\theta_2)$ is finitely presented.

This is true since K_1 is finitely generated and $\mathrm{im}(\theta_1)$ is $\mathbb{Z}_{\mathcal{V}}$-torsion-free and so the short exact sequence

$$0 \to K_1 \to R[H]^{d_1} \to \mathrm{im}(\theta_1) \to 0 \tag{28}$$

implies that the explicit criteria of Proposition 2.13(iii) (with M replaced by K_1) is satisfied.

Claim (ii) follows directly from claim (i). To prove claim (iii), we note that the stated conditions allow one, firstly, to apply Proposition 2.13(ii) with $M = N$ to deduce that N is finitely presented, and then Proposition 2.13(iii) with M taken to be the (finitely generated) kernel of any surjective homomorphism of $R[H]$-modules of the form $R[H]^a \to N$ (so that $R[H]^a/M$ is isomorphic to N) to deduce N is finitely 2-presented. Given this, claim (iii) follows directly from claim (ii). □

We next recall that a domain is said to be a 'finite conductor domain' if the intersection of any two principal ideals is finitely generated and that any coherent domain automatically has this property (cf. Glaz [14], but note that the concept was first considered by Dobbs in [11]). To prove that R is not a finite conductor domain, and hence not coherent, it is therefore enough to exhibit two non-zero principal ideals I_1 and I_2 of R such that the ideal $I_1 \cap I_2$ is not finitely generated. For this purpose, we set $I_1 := J = R \cdot \omega_0$ and $I_2 := R \cdot x$ for an element x of I whose image under the homomorphism $\kappa : I \to \mathbb{Z}_p/\mathbb{Z}_{\mathcal{V}}$ from Proposition 2.1(ii) (with $n = 0$) has infinite order. Then there exists a composite isomorphism of R-modules

$$J/(J \cap (R \cdot x)) \cong (J + (R \cdot x))/J \xrightarrow{\kappa} R \cdot \kappa(x) \cong \mathbb{Z}_{\mathcal{V}}.$$

In particular, since J is isomorphic to R, this isomorphism implies that $J \cap (R \cdot x)$ is isomorphic, as an R-module, to I and hence is not finitely generated, as required.

To prove that R has weak Krull dimension two (in the sense of [40]), it is enough to show that the maximal integer m for which there exists a chain

$$(0) \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_m$$

of finitely generated prime ideals of R is equal to 2. If, firstly, $\mathfrak{p}_1$ contains ϖ_n for some n, then it is a maximal ideal (by Theorem 2.18(iii)) and so, in this case $m = 1$. We can therefore assume that $\mathfrak{p}_1$ contains no element of the form ϖ_n and hence, by Theorem 2.18(iv), that $\mathfrak{p}_1 = R \cap \Lambda f_1$ for an irreducible distinguished polynomial f_1.

We claim next that $\mathfrak{p}_2$ must contain an element of the form ϖ_n. Indeed, if this is not true, then $\mathfrak{p}_2 = R \cap \Lambda f_2$ for an irreducible distinguished polynomial f_2 and the implied inclusion $\Lambda f_1 \subsetneq \Lambda f_2$ is not possible. Hence $\mathfrak{p}_2$ contains an element of the form ϖ_n and then, just as above, $\mathfrak{p}_2$ is maximal and so $m = 2$, as required.

Before proceeding we recall that, for each pair of non-negative integers n and d, Costa defines an '(n, d)-domain' to be an integral domain in which every finitely n-presented module has projective dimension at most d (for details see [7, §1]). For convenience, we shall further say that an (n,d)-domain is 'strict' if it is neither an $(n-1,d)$-domain nor an $(n,d-1)$-domain.

Until further notice, we now assume both that $\mathcal{V}=\varnothing$ and p is not exceptional and use this hypothesis to show that R is a $(2,2)$-domain. To do this we must therefore show that, if H is trivial, then in any exact sequence of the form (27), the R-module K_1 is projective.

As a first step, we show $K_2 := \ker(\theta_2)$ is a projective R-module (and hence that R is 2-regular in the sense of [12]). To do this we consider the short exact sequence

$$0 \to K_2 \to R^{d_2} \xrightarrow{\theta_2} K_1 \to 0. \tag{29}$$

Then, since K_1 is a finitely generated submodule of R^{d_1} (and hence pro-discrete), and the quotient module R^{d_1}/K_1 is isomorphic to the torsion-free module $\mathrm{im}(\theta_1) \subseteq R^{d_0}$, the argument establishing the final assertion of Proposition 2.13(iii) implies that the natural inverse system $(\varrho_n(K_2))_n$ is pro-discrete and that K_2 is its limit. Since the argument proving Proposition 2.13(ii) also shows that the quantities $\mu_{R_n}(\varrho_n(K_2))$ are bounded independently of n, Theorem 2.11(iii) implies it is enough for us to show that, for each n, the R_n-module $\varrho_n(K_2)$ is projective.

Now, since $\varrho_n(K_2)$ is torsion-free, and Γ_n is a p-group, one knows that $\varrho_n(K_2)$ is a projective R_n-module if and only if $\varrho_n(K_2)_p$ is a free $R_{n,p}$-module. To show this we set $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p]]$ and consider the Λ-module $\widehat{K_2}^p := \varprojlim_n K_{2,(n),p} = \varprojlim_n \varrho_n(K_2)_p$.

This Λ-module is a submodule of Λ^{d_2} and hence is both finitely generated and such that $\big(\widehat{K_2}^p\big)^\Gamma$ vanishes. In particular, since Proposition 2.13(i) implies that the Γ-coinvariants of $\widehat{K_2}^p$ identifies with the $\mathbb{Z}_p$-free module $K_{2,(0),p} = \varrho_0(K_2)_p$, a classical result of Iwasawa theory (cf. [36, Prop. (5.3.19)(ii)]) implies $\widehat{K_2}^p$ is a free Λ-module. This in turn implies that each $R_{n,p}$-module $\varrho_n(K_2)_p = K_{2,(n),p} = R_{n,p} \otimes_\Lambda \widehat{K_2}^p$ is free, as required.

Now, to prove that the finitely generated R-module K_1 is projective, it suffices (by another application of Proposition 2.13(i) and Theorem 2.11(iii)) to show, for every n, that the R_n-module $K_{1,(n)}$ is projective. To do this we note that the exact sequence (29) induces an exact sequence of R_n-modules

$$0 \to \varrho_n(K_2) \to R_n^{d_2} \to K_{1,(n)} \to 0.$$

Thus, since $\varrho_n(K_2)$ is a projective R_n-module, the Γ_n-module $K_{1,(n)}$ is cohomologically-trivial. To prove $K_{1,(n)}$ is a projective R_n-module, it is thus enough to show that the (finite) $\mathbb{Z}$-torsion subgroup $\mathcal{T}_n$ of $K_{1,(n)}$ vanishes.

To prove this we note that, since $\mathrm{im}(\theta_1)$ is pro-discrete, the argument of Proposition 2.13(ii) can be applied to the short exact sequence (28) to imply both that there is an isomorphism $\mathcal{T}_n \cong \mathsf{Tor}^1_R(R_n, \mathrm{im}(\theta_1))_{\rm tor}$, and also that the limit $\varprojlim_n \mathcal{T}_n$ vanishes.

Since $\mathrm{im}(\theta_1)^{\varpi_n=0}$ vanishes, the former isomorphism combines with the exact sequence (22) (with M replaced by $\mathrm{im}(\theta_1)$) to imply the order of $\mathcal{T}_n$ is prime-to-p and this fact then implies that, for each $n' > n$, the map $\mathcal{T}_{n'} \to \mathcal{T}_n$ induced by the natural projection $K_{1,(n')} \to K_{1,(n)}$ is surjective. The vanishing of $\varprojlim_n \mathcal{T}_n$ therefore implies that each $\mathcal{T}_n$ vanishes, as required.

This proves that R is a $(2,2)$-domain, and so, to complete the proof of Theorem 1.1, we are reduced to showing that R is neither a $(1,2)$-domain nor a $(2,1)$-domain. Since the latter property is satisfied for all primes p and finite sets $\mathcal{V}$ not containing p, we state it as a separate result.

Proposition 2.21. *R is neither a $(1,2)$-domain nor a $(2,1)$-domain.*

Proof. To verify R is not a $(1,2)$-domain we must show that there exists a finitely presented R-module that does not have projective dimension at most two.

Since R is not coherent, we fix a finitely generated ideal I of R that is not finitely presented. Then the quotient R/I is finitely presented but not finitely 2-presented. Suppose now that R/I has projective dimension at most two, and hence that I has projective dimension one. Then, in any exact sequence of R-modules of the form $0 \to K \to R^d \to I \to 0$, the module K is projective but cannot be finitely generated. Since R is a domain, the rank of K is equal to the dimension of the space it spans over the quotient field of R, and hence is at most d. Since any projective R-module of finite rank is finitely generated by [44, Prop. 1.3], we therefore have a contradiction.

To prove R is not a $(2,1)$-domain, we fix a non-trivial R-module M that is finite and of p-power order. Then, for any sufficiently large n, one has $M = M^{\varpi_n=0}$ (by [36, Prop. 5.3.14(ii)]) and so M is an R_n-module (by Lemma 2.5(i)). From Remark 2.15, it follows that M is finitely 2-presented.

Suppose now that M has projective dimension one. Then there exists an exact sequence of R-modules $0 \to K \to R^d \to M \to 0$ in which K is both finitely generated and projective. We can then consider the induced exact sequence

$$0 \to \mathsf{Tor}^1_R(M, R_n) \to K_{(n)} \to R^d_n \to M_{(n)} \to 0.$$

In particular, since the R_n-module $K_{(n)}$ is projective, this sequence implies that $\mathsf{Tor}^1_R(M, R_n)$ is $\mathbb{Z}_\mathcal{V}$-torsion-free. Then, since $\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}$ is uniquely p-divisible, and M is a finite p-group, the exact sequence (22) implies that the natural map $M = M^{\varpi_n=0} \to \mathsf{Tor}^1_R(M, R_n)$ is bijective, and hence that M vanishes. This contradiction completes the proof. □

Remark 2.22. Following Theorem 1.1, it would be interesting to know if R has Krull dimension two or is 'regular' (in the sense that every finitely generated ideal has a projective resolution of finite length). As yet, we have not been able to decide either of these issues.

3. Some auxiliary results

In this section we collect some auxiliary results and, in particular, derive several useful technical consequences of the algebraic methods that were developed above in order to prove Theorem 1.1.

3.1. **Mixed-characteristic Iwasawa algebras.** In the first result we say that a ring A is *semi-hereditary* if every finitely generated ideal is projective (and we note that it is straightforward to verify that any such ring is coherent).

Lemma 3.1. *Let ℓ be a rational prime different from p. Then the ring $\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ is semi-hereditary and of infinite Krull dimension.*

Proof. At the outset we write $S_n := \mathbb{Z}_\ell[\Gamma_n]$ and note that there is a decomposition

$$S[[\mathbb{Z}_p]] = \varprojlim_{n\geq 0} S_n \cong \prod_{n\geq 0} S_n e_n$$

Here we set $e_0 := 1$ and, for $n > 0$, write e_n for the idempotent $e_{n,n-1}$ of S_n that is defined in the proof of Theorem 2.11(ii).

For each natural number n the ring $S_n e_n$ is a finite direct product of integral extensions of $\mathbb{Z}_\ell$ and is therefore both semi-hereditary and of positive Krull dimension. The first claim of the Lemma thus follows from the fact that an arbitrary product of semi-hereditary rings is semi-hereditary, which itself is easily verified. The second claim, on the other hand, follows via an application of [13, Th. 3.3]. □

3.2. **$\mathbb{Z}$-torsion modules.** We next consider R-modules in which every element is annihilated by an integer that is prime-to-p. To do this we say that an R-module has *flat dimension* n for some natural number n if it admits a minimal resolution by flat R-modules of length n.

Lemma 3.2. *Assume both that $\mathcal{V} = \varnothing$ and p is not exceptional, and let Q be an R-module in which every element is annihilated by an integer that is prime-to-p. Then Q has flat dimension one and hence, if finitely presented, has projective dimension at most one.*

Proof. We claim that it suffices to consider the case that Q is finitely presented. To see this note, firstly, that Q is the direct limit of finitely generated R-modules that have finite exponent prime-to-p. In particular, since the functor $\mathsf{Tor}^1_R(-,-)$ commutes with direct limits (so that a direct limit of modules of flat dimension at most one has flat dimension at most one), we are reduced to considering the case that Q is finitely generated over R and of exponent m prime-to-p. Next observe that, by an induction on the number of generators of Q, one can construct a finite filtration of Q by submodules, the graded quotients of which each isomorphic to a module of the form R/K for some ideal K containing m. In particular, Q has flat dimension one as soon as any module of this form enjoys the same property.

As before, we may decompose K as the direct limit $K = \varinjlim_{n\in\mathbb{N}} K(n)$ of the finitely generated ideals contained therein. Note that there exists some $i \in \mathbb{N}$ such that for all $j \geq i$ one has that $m \in K(j)$. If it were indeed the case that each $R/K(j)$ has projective dimension one then $K(j)$ would be projective so that K is a direct limit of projective R-modules of finite rank. We would then deduce that K is a flat module so that R/K has flat dimension one.

We may therefore suppose that Q is a finitely presented R-module. In this case, Q is naturally a finitely presented module over the algebra

$$R/m = \varprojlim_{n\geq 0}\big(R_n/m\big) \cong \prod_{n\geq 0}\big((R_n e_n)/m\big),$$

where we set $e_0 := 1$ and, for $n > 0$, write e_n for the idempotent $e_{n,n-1}$ of $R_n[1/p]$ defined in (12). In particular, since for each $n > 0$ one has

$$I_n \cdot \big((R_a e_a)/m\big) = \begin{cases} \big((R_a e_a)/m\big), & \text{if } a > n,\\ 0, & \text{otherwise,}\end{cases}$$

the induced decomposition $Q = \prod_{n\geq 0} e_n Q$ implies that Q is pro-discrete.

We next fix an exact sequence of R-modules $0 \to M \xrightarrow{\phi} R^d \to Q \to 0$. Then, for each natural number n, this sequence induces exact sequences of R_n-modules

$$0 \to Q[1]_n \to M_{(n)} \xrightarrow{\phi_{(n)}} K_n \to 0, \quad 0 \to K_n \xrightarrow{\subseteq} R_n^d \to Q_{(n)} \to 0,$$

in which we set $Q[1]_n := \mathsf{Tor}^1_R(Q, R_n)$ and $K_n := \mathrm{im}(\phi_{(n)})$. In particular, since p acts invertibly on both $Q[1]_n$ and $Q_{(n)}$, these sequences imply the following: K_n is cohomologically-trivial over Γ_n and hence a projective R_n-submodule of R_n^d; the first exact sequence is split and so the natural maps $R_{n-1} \otimes_{R_n} Q[1]_n \to Q[1]_{n-1}$ and $R_{n-1} \otimes_{R_n} K_n \to K_{n-1}$ are bijective. In particular, since Q is pro-discrete, we may take the limit over n of the second exact sequence above to obtain an exact sequence of R-modules

$$0 \to K \to R^d \to Q \to 0,$$

with $K := \varprojlim_n K_n$. Then, since Q is pro-discrete, the argument following diagram (17) (with the role of M now played by Q) combines with the proof of Theorem 1.1 given in §2.5 to imply that K is a pro-discrete projective R-module. Given this, the last displayed sequence implies that the R-module Q has projective dimension at most one, as required. □

3.3. **Fitting ideals of finitely ∞-presented modules.** The following result clarifies, under suitable hypotheses, the connection between the initial Fitting ideal of a finitely ∞-presented $R[H]$-module M and the corresponding Fitting ideals of the associated $R_n[H]$-modules $M_{(n)}$.

Proposition 3.3. *Let H be a finite abelian group, d a natural number and*

$$P \to R[H]^d \to M \to 0 \tag{30}$$

an exact sequence of $R[H]$-modules where P is the limit of a pro-discrete system $(P_n)_n$ in which each $R_n[H]$-module P_n is locally-free of rank d. Then the following claims are valid.

(i) *The $R[H]$-module M is finitely ∞-presented.*

(ii) *The initial Fitting ideal $\mathrm{Fit}^0_{R[H]}(M)$ is finitely generated and lies in an exact sequence of $R[H]$-modules*

$$0 \to \mathrm{Fit}^0_{R[H]}(M) \to \varprojlim_n \mathrm{Fit}^0_{R_n[H]}(M_{(n)}) \xrightarrow{\kappa_M} \varprojlim{}^1_n \mathrm{Ann}_{R_n[H]}\big(\mathrm{Fit}^0_{R_n[H]}(M_{(n)})\big) \to 0 \tag{31}$$

in which the second arrow is the natural inclusion (of ideals of $R[H]$) and κ_M is a natural connecting homomorphism.

(iii) *If N is any pro-discrete ideal of $R[H]$ for which every $R_n[H]$-module $N_{(n)}$ is invertible, then there exists a canonical exact sequence of $R[H]$-modules*

$$0 \to N \to \varprojlim_n \varrho_n(N) \xrightarrow{\kappa_N} \varprojlim{}^1_n \mathrm{Ann}_{R_n[H]}\big(\varrho_n(N)\big) \to 0 \tag{32}$$

in which the second arrow is the natural inclusion of ideals.

Proof. We set $\mathbb{A} := R[H]$ and $\mathbb{A}_n := R_n[H]$ for each n. If S denotes either $\mathbb{A}$ or $\mathbb{A}_n$, then for each finitely-presented S-module N we respectively write $\mathcal{F}(N)$ and $\mathcal{F}_n(N)$ for $\mathrm{Fit}^0_S(N)$.

We note first that the given hypotheses on P combine with Theorem 2.11(ii) to imply it is a finitely generated pro-discrete $\mathbb{A}$-module and then Proposition 2.13(i) implies that, for every n, the $\mathbb{A}_n$-module $P_{(n)}$ identifies with P_n.

To prove claim (i), Proposition 2.20(ii) reduces us to showing that P is finitely presented. In view of Proposition 2.13(ii), it is therefore enough to show that, for every m, the group $\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_m, P)$ vanishes.

To show this we fix n and note that, for each integer $t > 1$, Theorem 2.11(ii) implies the existence of a free $\mathbb{A}$-submodule $P' = P'(n,t)$ of P that has rank d and is such that the natural map $P'_{(n)} \xrightarrow{\nu} P_{(n)}$ is injective and has finite cokernel of order prime to t. Then the tautological exact sequence $0 \to P' \to P \to P/P' \to 0$ gives rise to an exact sequence

$$\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P') \to \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P) \xrightarrow{\nu'} \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P/P') \to P'_{(n)} \xrightarrow{\nu} P_{(n)}.$$

Since the first term in this sequence vanishes (as P' is free) and ν is injective, the map ν' is bijective. In addition, the $\mathbb{A}_n$-module $\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P/P')$ is annihilated by the image of $\mathcal{F}(P/P')$ under the projection map $\varrho_n : \mathbb{A} \to \mathbb{A}_n$. Thus, since $\varrho_n\big(\mathcal{F}(P/P')\big) = \mathcal{F}_n((P/P')_{(n)})$ (by standard functorial properties of Fitting ideals) and $(P/P')_{(n)}$ is isomorphic to the finite module $\mathrm{cok}(\nu)$, it follows that $\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P/P')$, and hence also $\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P)$, is annihilated by a natural number that is prime to t. Since t can be chosen arbitrarily, this in turn implies that $\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, P)$ vanishes, as required to prove claim (i).

Turning to claim (ii), we note that, since M is finitely presented, the $\mathbb{A}$-ideal $\mathcal{F}(M)$ is finitely generated. To compute this ideal we fix n, t and $P' = P'(n,t)$ as above, write θ for the map $P \to \mathbb{A}^d$ in (30) and then M' for the cokernel of the composite map $P' \subseteq P \xrightarrow{\theta} \mathbb{A}^d$. We then claim that there is an equality

$$\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M')) = \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)})). \qquad (33)$$

To verify this we note that, since P' is free of rank d, $\mathcal{F}(M')$ is a principal $\mathbb{A}$-ideal and hence, after choosing a generator λ, that there exists an exact sequence of $\mathbb{A}$-modules $0 \to \mathbb{A} \xrightarrow{\times\lambda} \mathbb{A} \to \mathbb{A}/\mathcal{F}(M') \to 0$. This exact sequence induces an identification

$$\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M')) = \ker\big(\mathbb{A}_n \xrightarrow{\varrho_n(\lambda)} \mathbb{A}_n\big) = \{r \in \mathbb{A}_n : r \cdot \varrho_n(\lambda) = 0\} = \mathrm{Ann}_{\mathbb{A}_n}(\varrho_n(\mathcal{F}(M')),$$

where the last equality is valid since $\varrho_n(\mathcal{F}(M')) = \mathbb{A}_n \cdot \varrho_n(\lambda)$. To deduce (33), it is then enough to note that $\varrho_n(\mathcal{F}(M')) = \mathcal{F}_n(M'_{(n)})$.

We note next that $\mathcal{F}(M') \subseteq \mathcal{F}(M)$ and hence that there exists a tautological short exact sequence $0 \to \mathcal{F}(M)/\mathcal{F}(M') \to \mathbb{A}/\mathcal{F}(M') \to \mathbb{A}/\mathcal{F}(M) \to 0$. Taken in conjunction with (33), this sequence gives rise to an exact sequence of $\mathbb{A}_n$-modules

$$\begin{aligned}\mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, \mathcal{F}(M)/\mathcal{F}(M')) &\xrightarrow{\alpha'} \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)})) \\ &\xrightarrow{\alpha} \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M)) \to \big(\mathcal{F}(M)/\mathcal{F}(M')\big)_{(n)}. \qquad (34)\end{aligned}$$

Now, since P/P' is the kernel of the surjective map $M' \to M$, a standard multiplication property of Fitting ideals implies that the quotient $\mathcal{F}(M)/\mathcal{F}(M')$ is annihilated by the ideal $\mathcal{F}(P/P')$. The two end terms in the above sequence are therefore annihilated by the ideal $\varrho_n(\mathcal{F}(P/P')) = \mathcal{F}_n((P/P')_{(n)})$ of $\mathbb{A}_n$. In particular, since the module $(P/P')_{(n)}$ is finite and of order prime to t (as noted above), we can deduce that the index c of $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)}))$ in $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M_{(n)}))$ is finite and prime to t, and also that both end terms in the exact sequence (34) are annihilated by a natural number c' that is prime to t.

Since $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)}))$ is $\mathbb{Z}_{\mathcal{V}}$-free, its submodule $\ker(\alpha) = \mathrm{im}(\alpha')$ is therefore zero and so we can identify $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)}))$ with its image under α.

In addition, if we now repeat this argument after replacing t by the product cc', we can deduce the existence of a $\mathbb{A}$-module M'' with the property that

$$\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M_{(n)})) = \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)})) + \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M''_{(n)})) = \mathsf{Tor}^1_{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M)).$$

Taking account of this equality, the tautological short exact sequence

$$0 \to \mathcal{F}(M) \to \mathbb{A} \to (\mathbb{A}/\mathcal{F}(M)) \to 0$$

combines with the identification $\varrho_n(\mathcal{F}(M)) = \mathcal{F}_n(M_{(n)})$ to induce, for every n, a canonical short exact sequence of $\mathbb{A}_n$-modules

$$0 \to \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M_{(n)})) \to \mathcal{F}(M)_{(n)} \to \mathcal{F}_n(M_{(n)}) \to 0.$$

At this point, we note that $\mathcal{F}(M)$ is a finitely generated submodule of $\mathbb{A}$ and so is pro-discrete (by Proposition 2.13(iii)). By passing to the limit over n of the above exact sequences, we therefore obtain an exact sequence of the form (31), as required to prove claim (ii).

To prove claim (iii) we use the fact that $N = \varprojlim_n N_n$ for a pro-discrete family $(N_n)_n$ of $\mathbb{A}_n$-modules. In particular, for every n the surjectivity of the map $N_{(n)} \to N_n$ combines with Remark 2.16 and the fact $N_{(n)}$ is invertible to imply $\mu_{\mathbb{A}_n}(N_n) \le 2$. The ideal N is therefore finitely generated (by Theorem 2.11(ii)) and hence also such that the canonical map $N_{(n)} \to N_n$ is bijective (by Proposition 2.13(i)).

It follows that the quotient module $M = \mathbb{A}/N$ has a presentation of the form (30) (with $P = N$ and $d = 1$). In addition, one has $\mathcal{F}(\mathbb{A}/N) = N$ and, for each n, also $\mathcal{F}_n((\mathbb{A}/N)_{(n)}) = \mathcal{F}_n(\mathbb{A}_n/\varrho_n(N)) = \varrho_n(N)$ and so the claimed exact sequence is just a special case of that in claim (ii). □

Remark 3.4. The derived limit $\varprojlim^1_n \mathrm{Ann}_{R_n[H]}\big(\varrho_n(N)\big)$ that occurs in Proposition 3.3(iii) can be non-zero. For example, if H is trivial and, for any non-negative integer n, one sets $N := J_n = \varpi_n R$, then it can be checked that the short exact sequence (32) recovers that of Proposition 2.1(ii).

Remark 3.5. In the setting of Proposition 3.3, assume x is a non-zero divisor of $R[H]$ with $\mathrm{Fit}^0_{R_n[H]}(M_{(n)}) = R_n[H] \cdot \varrho_n(x)$ for every n. Then, in this case, a comparison of the exact sequence in claim (ii) with that in claim (iii) for the (free) $R[H]$-module N generated by x, suggests that one might expect an equality $\mathrm{Fit}^0_{R[H]}(M) = R[H] \cdot x$. However, even if the $R[H]$-module P is free (of rank d), p is not exceptional, and $\mathcal{V} = \varnothing$, it seems unlikely that this is always true. Indeed, in this case, one has $\mathrm{Fit}^0_{R[H]}(M) = R[H] \cdot \delta$ with δ the determinant of any matrix in $\mathrm{M}_d(R[H]^d)$ that represents (with respect to any choice of bases of P and $R[H]^d$) the map $P \to \mathbb{A}^d$ in (30), and $R[H] \cdot \delta = R[H] \cdot x$ if and only if there exists a element u of $R[H]^\times = \varprojlim_n R_n[H]^\times$ with $\delta = u \cdot x$. For any given n, however, the equality $R_n[H] \cdot \varrho_n(\delta) = R_n[H] \cdot \varrho_n(x)$ does not itself imply that $\varrho_n(\delta)$ and $\varrho_n(x)$ differ by a unit in $R_n[H]^\times$ since the $R_n[H]$-modules generated by $\varrho_n(\delta)$ and $\varrho_n(x)$ need not be free.

To end this section, we make a technical observation about derived limits (of the sort that occur in Proposition 3.3) that will be useful in the sequel.

Lemma 3.6. *Let $(A_n)_n$ be an inverse system of finitely generated $\mathbb{Z}_\mathcal{V}$-modules. Then the derived limit $\varprojlim^1_n A_n$ is divisible, and its torsion subgroup is supported on primes $\ell \notin \mathcal{V}$ for which $\varprojlim_n(\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}})$ is non-zero.*

Proof. Since each torsion subgroup $A_{n,\mathrm{tor}}$ is finite, the Mittag-Leffler criterion implies that the derived limits $\varprojlim^i_n A_{n,\mathrm{tor}}$ vanish for both $i = 1$ and $i = 2$. By passing to the limit over n of the tautological short exact sequences $0 \to A_{n,\mathrm{tor}} \to A_n \to A_{n,\mathrm{tf}} \to 0$, one deduces that the natural map $\varprojlim^1_n A_n \to \varprojlim^1_n A_{n,\mathrm{tf}}$ is bijective.

Then, for each prime $\ell \notin \mathcal{V}$ there is a short exact sequence

$$0 \to A_{n,\mathrm{tf}} \to \mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}} \to (\mathbb{Z}_\ell/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}} \to 0.$$

In particular, since each module $\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}}$ is compact, and inverse limits are exact on the category of compact modules, by passing to the limit over n of these exact sequences, we obtain an exact sequence

$$\varprojlim_n\big(\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}}\big) \to \varprojlim_n\big((\mathbb{Z}_\ell/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}}\big) \to \varprojlim^1_n A_{n,\mathrm{tf}} \to 0.$$

Now, since the quotient group $\mathbb{Z}_\ell/\mathbb{Z}_\mathcal{V}$ is uniquely ℓ-divisible, the second term in this sequence is also uniquely ℓ-divisible. The exact sequence therefore implies that the group $\varprojlim^1_n A_n = \varprojlim^1_n A_{n,\mathrm{tf}}$ is ℓ-divisible, respectively uniquely ℓ-divisible if $\varprojlim_n\big(\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} A_{n,\mathrm{tf}}\big)$ vanishes. Since ℓ is an arbitrary prime outside $\mathcal{V}$, the claimed result follows. □

4. Iwasawa Theory over $\mathbb{Z}[[\mathbb{Z}_p]]$

In this section we continue to denote by $\mathcal{V}$ a fixed finite set of rational primes that does not contain p and use R and R_n to denote the rings $\mathbb{Z}_\mathcal{V}[[\mathbb{Z}_p]]$ and $\mathbb{Z}_\mathcal{V}[\mathbb{Z}/(p^n)]$.

We shall apply the algebraic results obtained above to study $\mathbb{Z}_\mathcal{V}[[\mathbb{Z}_p]]$-modules arising from the inverse limits of ideal class groups (in §4.1), of integral dual Selmer groups for $\mathbb{G}_m$ (in §4.2) and of universal norm groups of units (in §4.3). In this way we shall prove Theorem 1.2 by specialisation to the case $\mathcal{V} = \varnothing$ and also, in suitable cases, establish a precise link between the Fitting ideals over $\mathbb{Z}_\mathcal{V}[[\mathbb{Z}_p]]$ of dual Selmer groups and Stickelberger elements (see Theorem 4.5). We note, in particular, that the latter result suggests a possible 'main conjecture of integral Iwasawa theory' that refines the classical main conjecture (see Remark 4.6 and Question 4.7).

For a finite extension F/K of global fields, we use the following notation. For a set of places U of K we write U_F for the set of places of F above those in U, $Y_{F,U}$ for the free abelian group on the set U_F and $X_{F,U}$ for the kernel of the homomorphism $Y_{F,U} \to \mathbb{Z}$ that sends each place in U_F to 1. If U is finite, non-empty and contains the set $S_\infty(K)$ of archimedean places (so that $S_\infty(K) = \varnothing$ if K is a function field), then we write $\mathcal{O}_{F,U}$ for the ring of U_F-integers of F and

$$\mathrm{Cl}_U(\mathcal{O}_F) = \mathrm{Cl}_{U,\mathcal{V}}(\mathcal{O}_F)$$

for the maximal subgroup of the ideal class group of $\mathcal{O}_{F,U}$ whose order is supported on those primes outside $\mathcal{V}$. We note, in particular, that, if $\mathcal{V} = \varnothing$, then $\mathrm{Cl}_U(\mathcal{O}_F)$ is equal to the ideal class group of $\mathcal{O}_{F,U}$ for every F.

4.1. **Class group structures and the proof of Theorem 1.2.** Fix a $\mathbb{Z}_p$-extension K_∞ of a global field K and a set S of places of K containing $S_\infty(K)$.

4.1.1. *The R-module $\mathrm{Cl}_S(K_\infty)$ is pro-discrete.* For each natural number n the homomorphism $\mathrm{Cl}_S(K_n) \to \mathrm{Cl}_S(K_{n-1})$ induced by field-theoretic norm restricts to give, for each prime ℓ, a map $\mathrm{Cl}_S(K_n)_\ell \to \mathrm{Cl}_S(K_{n-1})_\ell$. For each such ℓ one therefore obtains an R-submodule of $\mathrm{Cl}_S(K_\infty) := \varprojlim_n \mathrm{Cl}_S(K_n)$ by setting $\mathrm{Cl}_S(K_\infty)_\ell := \varprojlim_n \mathrm{Cl}_S(K_n)_\ell$.

These submodules in turn give rise to a canonical direct sum decomposition of R-modules

$$\mathrm{Cl}_S(K_\infty) = \mathrm{Cl}_S(K_\infty)_p \oplus \mathrm{Cl}_S(K_\infty)'. \tag{35}$$

Here we have set

$$\mathrm{Cl}_S(K_\infty)' := \varprojlim_n \mathrm{Cl}_S(K_n)' = \prod_{\ell \neq p} \mathrm{Cl}_S(K_\infty)_\ell,$$

where, for each n, we write $\mathrm{Cl}_S(K_n)'$ for the complement $\bigoplus_{\ell\neq p} \mathrm{Cl}_S(K_n)'$ of $\mathrm{Cl}_S(K_n)_p$ in $\mathrm{Cl}_S(K_n)$.

To prove $\mathrm{Cl}_S(K_\infty)$ is pro-discrete, it is therefore enough to prove that the modules $\mathrm{Cl}_S(K_\infty)_p$ and $\mathrm{Cl}_S(K_\infty)'$ are each pro-discrete.

The key point about $\mathrm{Cl}_S(K_\infty)_p$ is that it is a finitely generated module over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$ and that, for each n, Lemma 2.5(i) implies that $\mathrm{Cl}_S(K_\infty)_{p,(n)}$ identifies with the quotient of $\mathrm{Cl}_S(K_\infty)_p$ by $\varpi_n(\mathrm{Cl}_S(K_\infty)_p)$. In this case therefore, the bijectivity of the natural map

$$\mathrm{Cl}_S(K_\infty)_p \to \varprojlim_n \mathrm{Cl}_S(K_\infty)_{p,(n)}$$

follows from a standard property of finitely generated Λ-modules.

To deal with the module $\mathrm{Cl}_S(K_\infty)'$ it is enough to show that for each $n > 0$ the homomorphism

$$\overline{\eta_n} : R_{n-1} \otimes_{R_n} \mathrm{Cl}_S(K_n)' \to \mathrm{Cl}_S(K_{n-1})'$$

that is induced by the norm map $\eta_n : \mathrm{Cl}_S(K_n)' \to \mathrm{Cl}_S(K_{n-1})'$ is bijective.

To prove this we write ι_n for the natural inflation map $\iota_n : \mathrm{Cl}_S(K_{n-1})' \to \mathrm{Cl}_S(K_n)'$ and note that, for every x in $\mathrm{Cl}_S(K_{n-1})'$ one has $px = \eta_n(\iota_n(x))$.

Since the order of $\mathrm{Cl}_S(K_{n-1})'$ is prime-to-p, this equality implies both that ι_n is injective and that η_n, and hence also $\overline{\eta_n}$, is surjective. In particular if we use ι_n to regard $\mathrm{Cl}_S(K_{n-1})'$ as a submodule of $\mathrm{Cl}_S(K_n)'$, then η_n is induced by the action of $\sum_{g\in \mathrm{Gal}(K_n/K_{n-1})} g$. Since the order of $\mathrm{Cl}_S(K_n)'$ is prime-to-p, it follows that $\ker(\eta_n) = \varpi_{n-1}(\mathrm{Cl}_S(K_n)')$ and hence that $\overline{\eta_n}$ is bijective, as required.

4.1.2. *The R-module $\mathrm{Cl}_S(K_\infty)$ is torsion.* The natural projection map

$$\varprojlim_n \mathrm{Cl}_{S,\varnothing}(K_n) \to \varprojlim_n \mathrm{Cl}_{S,\mathcal{V}}(K_n)$$

is surjective. Since $\mathbb{Z}[[\mathbb{Z}_p]]$ is contained in $\mathbb{Z}_{\mathcal{V}}[[\mathbb{Z}_p]]$, it is therefore enough to prove the claimed assertion in the case $\mathcal{V} = \varnothing$ (which we henceforth assume).

Then, since R is a domain, the module $\mathrm{Cl}_S(K_\infty)$ is torsion if there exists a non-zero element of R that annihilates it.

To prove the existence of such an element, we fix a non-empty finite set of places U of K that comprises the union of S, $S_\infty(K)$, all places that ramify in K_∞ and, in the case that K is a function field, $S = \varnothing$ and K_∞/K is unramified, a choice of place that does not split completely in K_∞. We recall that, for each n, the U-truncated transpose Selmer group $\mathcal{S}_n^{\mathrm{tr}} = \mathcal{S}_{U,\varnothing}^{\mathrm{tr}}(\mathbb{G}_m/K_n)$ defined in [5, §2.2] lies in a canonical exact sequence of R_n-modules

$$0 \to \mathrm{Cl}_U(K_n) \to \mathcal{S}_n^{\mathrm{tr}} \to X_{K_n,U} \to 0, \tag{36}$$

and is such that there exists a natural isomorphism $R_{n-1} \otimes_{R_n} \mathcal{S}_n^{\mathrm{tr}} \cong \mathcal{S}_{n-1}^{\mathrm{tr}}$.

By a standard property of Fitting ideals, the latter isomorphism implies that, for every non-negative integer a, the natural projection map $\varrho_{n,n-1} : R_n \to R_{n-1}$ restricts to give an equality

$$\varrho_{n,n-1}(\mathrm{Fit}^a_{R_n}(\mathcal{S}_n^{\mathrm{tr}})) = \mathrm{Fit}^a_{R_{n-1}}(\mathcal{S}_{n-1}^{\mathrm{tr}}). \tag{37}$$

To use this fact, we write Y_n^∞ for the free abelian group on the set $S_\infty(K_n)$. We then consider the submodule $\mathcal{S}_{n,1}^{\mathrm{tr}}$ of $\mathcal{S}_n^{\mathrm{tr}}$ that is defined by the exact sequence

$$0 \to \mathcal{S}_{n,1}^{\mathrm{tr}} \xrightarrow{\subseteq} \mathcal{S}_n^{\mathrm{tr}} \to Y_n^\infty \to 0.$$

Here the (surjective) third arrow is the composite $\mathcal{S}_n^{\mathrm{tr}} \to X_{K_n,U} \to Y_n^\infty$, where the first map is from the sequence (36) and the second is the natural projection map. In particular, since Y_n^∞ is a free R_n-module of rank $r_K := |S_\infty(K)|$ (so that Y_n^∞ vanishes if K is a function field), the above sequence implies that $\mathrm{Fit}^{r_K}_{R_n}(\mathcal{S}_n^{\mathrm{tr}}) = \mathrm{Fit}^0_{R_n}(\mathcal{S}_{n,1}^{\mathrm{tr}})$ and hence, by (37) with $a = r_K$, that

$$\varrho_{n,n-1}(\mathrm{Fit}^0_{R_n}(\mathcal{S}_{n,1}^{\mathrm{tr}})) = \mathrm{Fit}^0_{R_{n-1}}(\mathcal{S}_{n-1,1}^{\mathrm{tr}}) \subseteq \mathrm{Ann}_{R_{n-1}}(\mathcal{S}_{n-1,1}^{\mathrm{tr}}).$$

As such, since

$$\varprojlim_n \mathrm{Fit}^0_{R_n}(\mathcal{S}_{n,1}^{\mathrm{tr}}) \subseteq \varprojlim_n \mathrm{Ann}_{R_n}(\mathcal{S}_{n,1}^{\mathrm{tr}}) \subseteq \mathrm{Ann}_R(\mathcal{S}_{\infty,1}^{\mathrm{tr}})$$

with $\mathcal{S}_{\infty,1}^{\mathrm{tr}} := \varprojlim_n \mathcal{S}_{n,1}^{\mathrm{tr}}$, it follows that the R-module $\mathcal{S}_{\infty,1}^{\mathrm{tr}}$ is torsion provided that $\mathrm{Fit}^0_{R_n}(\mathcal{S}_{n,1}^{\mathrm{tr}})$ is non-zero for any specific value of n. This is true, however, since if we choose n large enough to ensure that no place in $U \setminus S_\infty(K)$ is completely split in K_n, then the exact sequence

$$0 \to \mathrm{Cl}_U(K_n) \to \mathcal{S}_{n,1}^{\mathrm{tr}} \to X_{K_n,U\setminus S_\infty(K)} \to 0$$

(induced by (36)) implies that the $\mathbb{Q}[\mathrm{Gal}(K_n/K)]$-module spanned by $\mathcal{S}_{n,1}^{\mathrm{tr}}$ has no quotient isomorphic to $\mathbb{Q}[\mathrm{Gal}(K_n/K)]$.

Since this last exact sequence also implies, by passing to the limit over n, that the R-module $\mathrm{Cl}_U(K_\infty) := \varprojlim_n \mathrm{Cl}_U(K_n)$ is isomorphic to a submodule of $\mathcal{S}_{\infty,1}^{\mathrm{tr}}$, our argument has therefore shown that $\mathrm{Cl}_U(K_\infty)$ is a torsion R-module.

We next note that, for each n, there is an exact commutative diagram of R_n-modules

$$\begin{array}{ccccccc} Y_{K_n,U\setminus S} & \xrightarrow{\alpha_n} & \mathrm{Cl}_S(K_n) & \longrightarrow & \mathrm{Cl}_U(K_n) & \longrightarrow & 0 \\ {\scriptstyle \iota_n}\downarrow & & \downarrow & & \downarrow & & \\ Y_{K_{n-1},U\setminus S} & \xrightarrow{\alpha_{n-1}} & \mathrm{Cl}_S(K_{n-1}) & \longrightarrow & \mathrm{Cl}_U(K_{n-1}) & \longrightarrow & 0, \end{array}$$

in which ι_n is induced by restriction of places and the other vertical arrows are the natural norm maps. In particular, since the class groups are finite, passing to the limit over n in these diagrams gives rise both to a surjective homomorphism $\beta : \mathrm{Cl}_S(K_\infty) \to \mathrm{Cl}_U(K_\infty)$ and an exact sequence of R-modules

$$\varprojlim_n Y_{K_n,U\setminus S} \to \ker(\beta) \to \varprojlim^1_n \ker(\alpha_n)$$

in which the limits are taken with respect to the maps ι_n.

To complete the proof that $\mathrm{Cl}_S(K_\infty)$ is a torsion R-module, it is now enough to note that, since the decomposition subgroup in $\mathrm{Gal}(K_\infty/K)$ of each place in $U \setminus S$ is open, the two end terms in the last displayed sequence are annihilated by the element ϖ_m for any sufficiently large m.

4.1.3. *Finite-presentability and the generalized Gross-Kuz'min Conjecture.* The direct sum decomposition (35) reduces us to considering separately the finite-presentability of the R-modules $\mathrm{Cl}_S(K_\infty)_p$ and $\mathrm{Cl}_S(K_\infty)'$.

In particular, since each of these modules are pro-discrete (by the argument in §4.1.1), to prove finite generation, we are reduced by Theorem 2.11(ii) to considering conditions under which, for every n, the quantities $\mu_{R_n}(\mathrm{Cl}_S(K_\infty)_{p,(n)})$ and $\mu_{R_n}(\mathrm{Cl}_S(K_\infty)'_{(n)})$ are finite and bounded independently of n.

To consider p-primary parts we recall that $\mathrm{Cl}_S(K_\infty)_p$ is a finitely generated torsion module over $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$, and hence that, for each n, the module $\mathrm{Cl}_S(K_\infty)_{p,(n)}$ is a finitely generated module over $\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} R_n$. Since $\mathbb{Z}_p$ is not finitely generated over $\mathbb{Z}_\mathcal{V}$, it follows that $\mathrm{Cl}_S(K_\infty)_{p,(n)}$ is finitely generated over R_n if and only if it is finite and, if this is the case, then $\mu_{R_n}(\mathrm{Cl}_S(K_\infty)_{p,(n)})$ is at most the number of generators of the Λ-module $\mathrm{Cl}_S(K_\infty)_p$. In particular, since the structure theory of finitely generated torsion Λ-modules implies that $\mathrm{Cl}_S(K_\infty)_{p,(n)}$ is finite if and only if $\mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0}$ is finite, we have shown that

$$\mathrm{Cl}_S(K_\infty)_p \;\text{ is finitely generated over } R \iff \mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0} \;\text{ is finite for every } n. \tag{38}$$

To consider $\mathrm{Cl}_S(K_\infty)'$ we note that, for each $n > 0$, the decomposition (13) (with $Q_n = \mathrm{Cl}_S(K_n)'$) combines with the argument of §4.1.1 to imply there is a natural identification of R_n-modules

$$\begin{aligned}\mathrm{Cl}_S(K_n)' &= \bigoplus_{0\le n'<n} e_{n,n'}(\mathrm{Cl}_S(K_{n'})') \\ &= \bigoplus_{\ell\neq p}\bigoplus_{0\le n'<n} e_{n,n'}(\mathrm{Cl}_S(K_{n'})_\ell) \\ &\cong \bigoplus_{\ell\neq p}\bigoplus_{0\le n'<n} \mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell.\end{aligned} \tag{39}$$

Here the modules $\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell$ are as defined just before the statement of Theorem 1.2 and so the isomorphism arises because the explicit definition of the idempotents $e_{n,n'}$ implies that $e_{n,n'}(\mathrm{Cl}_S(K_{n'})_\ell)$ is naturally isomorphic to $\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell$.

For each $n' \geq 0$ we now write $\mathcal{O}_n$ for the base-change to $\mathbb{Z}_\mathcal{V}$ of the ring of integers of the number field $\mathbb{Q}(e^{2\pi i/p^{n'}})$. Then the action of R on $\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell$ factors through the surjective ring homomorphism $R_n \to \mathcal{O}_n$ that sends the image of γ in R_n to $e^{2\pi i/p^{n'}}$.

We next fix a rational prime ℓ not contained in $\mathcal{V}\cup\{p\}$. If we write $\prod_{i\in I}\mathfrak{l}_i$ for the prime decomposition of the ideal $\ell\mathcal{O}_n$, then for any large enough integer m one has

$$\begin{aligned}\mu_{R_n}\bigl(\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell\bigr) &= \mu_{\mathcal{O}_n}\bigl(\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell\bigr) \\ &= \max_{i\in I}\bigl(\mu_{\mathcal{O}_{n,\mathfrak{l}_i}}\bigl(\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell/\mathfrak{l}_i^m\bigr)\bigr) \\ &= \max_{i\in I}\bigl(\mu_{\mathcal{O}_n/\mathfrak{l}_i}\bigl(\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell/\mathfrak{l}_i\bigr)\bigr).\end{aligned} \tag{40}$$

Here $\mathcal{O}_{n,\mathfrak{l}_i}$ is the localisation of $\mathcal{O}_n$ at the prime ideal $\mathfrak{l}_i$, the second equality follows from the Chinese Remainder Theorem, and the last from Nakayama's Lemma.

Now, since each field $\mathcal{O}_n/\mathfrak{l}_i$ has cardinality $\ell^{m(n')}$ with $m(n')$ equal to the order of the image of ℓ in $\bigl(\mathbb{Z}/(p^{n'})\bigr)^\times$, one has

$$\mu_{\mathcal{O}_n/\mathfrak{l}_i}\bigl(\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell/\mathfrak{l}_i\bigr) = m(n')^{-1}\cdot\mathrm{rk}_\ell\bigl(\mathrm{Cl}_S(K_{n'},K_{n'-1})_\ell/\mathfrak{l}_i\bigr).$$

Taken in conjunction with (40) and the decomposition (39), this implies that

$$\begin{aligned}&\mathrm{Cl}_S(K_\infty)' \text{ is a finitely generated } R\text{-module}\\ &\iff m(n)^{-1}\cdot\mathrm{rk}_\ell\bigl(\mathrm{Cl}_S(K_n,K_{n-1})_\ell\bigr) \text{ is bounded independently of both } \ell \text{ and } n.\end{aligned} \tag{41}$$

We now assume $\mathrm{Cl}_S(K_\infty)$ is a finitely generated R-module (so that the respective conditions on the right hand sides of (38) and (41) are satisfied) and prove that it is finitely ∞-presented. By Proposition 2.20(ii), it is in fact enough for us to show $\mathrm{Cl}_S(K_\infty)$ is finitely 2-presented.

As a first step, we note that, since $\mathrm{Cl}_S(K_\infty)$ is pro-discrete, the last assertion of Proposition 2.13(ii) (with $M=\mathrm{Cl}_S(K_\infty)$) implies that it is finitely presented if and only if the quantities

$$\mu_R(\mathrm{Cl}_S(K_\infty)^{\varpi_n=0}) = \max\bigl\{\mu_{R_n}(\mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0}),\mu_{R_n}((\mathrm{Cl}_S(K_\infty)')^{\varpi_n=0})\bigr\}$$

are bounded independently of n.

In this regard, we note Lemma 2.10 (with $M=\mathrm{Cl}_S(K_\infty)'$) implies that the R_n-module $(\mathrm{Cl}_S(K_\infty)')^{\varpi_n=0}$ is isomorphic to $\mathrm{Cl}_S(K_\infty)'_{(n)}$, and hence that $\mu_{R_n}((\mathrm{Cl}_S(K_\infty)')^{\varpi_n=0})\le\mu_R(\mathrm{Cl}_S(K_\infty)')$.

To consider $\mu_{R_n}(\mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0})$, we note that the structure theory of Λ-modules implies the existence of an exact sequence of Λ-modules of the form

$$0\to M_1\to\mathrm{Cl}_S(K_\infty)_p\to M_2\oplus M_2' \tag{42}$$

in which M_1 is a finite p-group, M_2 is a finitely generated torsion-free $\mathbb{Z}_p$-module and M_2' is a finite direct sum of modules of the form $\Lambda/(p^i)$. In particular, since $(M_2')^{\varpi_n=0}$ vanishes and $\mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0}$ is (by assumption) finite, the above sequence induces an isomorphism $M_1^{\varpi_n=0}\cong\mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0}$. Since M_1 is finite, it is therefore clear that the quantities $\mu_{R_n}((\mathrm{Cl}_S(K_\infty)_p)^{\varpi_n=0})$ are bounded independently of n, as required.

We now know $\mathrm{Cl}_S(K_\infty)$ is finitely-presented, and so can fix an exact sequence of R-modules of the form

$$R^m\xrightarrow{\psi}R^n\to\mathrm{Cl}_S(K_\infty)\to 0.$$

We must prove that the finitely generated R-module $M := \mathrm{im}(\psi)$ is finitely-presented. To do this we note that, for each n, the above sequence induces isomorphisms

$$(R^n/M)^{\varpi_n=0}[p^\infty] \cong \mathrm{Cl}_S(K_\infty)^{\varpi_n=0}[p^\infty] = \mathrm{Cl}_S(K_\infty)_p^{\varpi_n=0}[p^\infty] \cong M_1^{\varpi_n=0},$$

for the module M_1 discussed above. In particular, since M_1 is finite, we can apply the criterion of Proposition 2.13 (iii) in order to deduce that M is finitely-presented, as required.

To complete the proof of Theorem 1.2, it is now enough to show that, if K is a function field, K_∞ is equal to the constant $\mathbb{Z}_p$-extension K_∞^{con} and $S = \varnothing$, then the conditions (38) and (41) are both satisfied, and hence that Theorem 1.2(i) is valid.

Write κ for the constant field of K. Then K is the function field of a smooth projective curve C over κ and, if J is the Jacobian of C, then for every n one has $\mathrm{Cl}(K_n) = J(\kappa_n)$, where κ_n is the field extension of κ of degree n. This description implies that $\mathrm{Cl}(K_n) = \mathrm{Cl}(K_m)^{\varpi_n=0}$ for every $m > n$, and hence that

$$\mathrm{Cl}(K_\infty)_p^{\varpi_n=0} = \big(\varprojlim_{m>n} \mathrm{Cl}(K_m)_p\big)^{\varpi_n=0} = \varprojlim_{m>n} \big(\mathrm{Cl}(K_m)_p^{\varpi_n=0}\big) = \varprojlim_{m>n} \mathrm{Cl}(K_n)_p = 0,$$

where the transition morphisms in the third limit are multiplication-by-p. In this case, therefore, the condition (38) is clearly satisfied.

In addition, if g is the genus of C, then for every n and every prime $\ell \neq p$, one has $\mathrm{rk}_\ell(J(\kappa_n)) \leq 2g$ and so the condition (41) is satisfied in this case. (A similar observation about ℓ-ranks in this context is made by Washington in [46, §II]).

This completes the proof of Theorem 1.2. □

Remark 4.1. Let K be a number field. Then the $\mathrm{Gal}(K_\infty^{\mathrm{cyc}}/K)$-invariants of $\mathrm{Cl}_{S(p)}(K_\infty^{\mathrm{cyc}})_p$ were first conjectured to be finite by Jaulent in [25] as a natural generalisation of the earlier conjecture [18, Conj. 1.15] of Gross and Kuz'min (the link between these conjectures is established by Kolster in [29, Th. 1.14], where the result is attributed to Kuz'min [31]). In general, the $\mathrm{Gal}(K_\infty/K)$-invariants of $\mathrm{Cl}_{S(p)}(K_\infty)_p$ are known to be finite in each of the following cases:
(i) K is abelian over $\mathbb{Q}$ (cf. Greenberg [17]).
(ii) K is abelian over an imaginary quadratic field and $K_\infty = K_\infty^{\mathrm{cyc}}$ (cf. Maksoud [34]).
(iii) K has at most two p-adic places and $K_\infty = K_\infty^{\mathrm{cyc}}$ (cf. Kleine [28]).
(iv) K is totally real and validates Leopoldt's conjecture at p (cf. Remark 1.6).

4.2. **Dual Selmer groups and 'main conjectures' over $\mathbb{Z}[[\mathbb{Z}_p]]$.** In this section we prove a result in the spirit of a 'main conjecture' over $\mathbb{Z}_{\mathcal{V}}[[\mathbb{Z}_p]]$ rather than $\mathbb{Z}_p[[\mathbb{Z}_p]]$.

For any abelian group A we consider the duals

$$A^\vee := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad A^* := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}),$$

(each endowed with the contragredient action of any Galois group that acts on A).

If A is finite, then for any set of rational primes Σ we write A_Σ and A^Σ for the maximal subgroups of A whose orders are only divisible by primes in Σ and outside Σ respectively.

For a commutative ring B we write $\mathsf{D}(B)$ for the derived category of B-modules and $\mathsf{D}^p(B)$ for the full triangulated subcategory of $\mathsf{D}(B)$ comprising perfect complexes.

4.2.1. *Torsion-restricted dual Selmer groups.* In this subsection we fix a set of rational primes Σ.

We also fix a finite abelian extension F of a global field K of group G, a finite non-empty set of places S of K that contains $S_\infty(K)$ and a finite set of places T of K that is disjoint from S. We write $\mathcal{O}_{F,S}^{\times,T}$ for the (finite index) submodule of $\mathcal{O}_{F,S}^{\times}$ comprising all elements that are congruent to 1 modulo all places in T_F and $\mathrm{Cl}_S^T(F) = \mathrm{Cl}_{S,\mathcal{V}}^T(F)$ for the maximal subgroup of the ray class group of the ring $\mathcal{O}_{F,S}$ modulo T_F whose order is supported on those primes outside $\mathcal{V}$.

We also write $\mathcal{S}_{S,T}(\mathbb{G}_m/F)$ for the base-change over $\mathbb{Z}$ to $\mathbb{Z}_\mathcal{V}$ of the (S,T)-relative integral dual Selmer group of $\mathbb{G}_m$ over F, as defined in [5, §2]. In particular, recalling that $\mathcal{S}_{S,T}(\mathbb{G}_m/F)_{\mathrm{tor}} = \mathrm{Cl}_S^T(F)^\vee$ (by [5, Prop. 2.2]), we consider the Σ-restricted Selmer group

$$\mathcal{S}_S^T(F)_\Sigma := \mathcal{S}_{S,T}(\mathbb{G}_m/F)/(\mathrm{Cl}_S^T(F)^\vee)^\Sigma.$$

Lemma 4.2. *The following claims are valid.*

(i) *There exists a canonical exact sequence*

$$0 \to (\mathrm{Cl}_S^T(F)_\Sigma)^\vee \to \mathcal{S}_S^T(F)_\Sigma \to (\mathcal{O}_{F,S}^{\times,T})^* \otimes_\mathbb{Z} \mathbb{Z}_\mathcal{V} \to 0.$$

In the following assertions we assume that Σ contains all primes that divide the order of G.

(ii) *The tautological exact sequence*

$$0 \to (\mathrm{Cl}_S^T(F)^\vee)^\Sigma \to \mathcal{S}_S^T(F) \to \mathcal{S}_S^T(F)_\Sigma \to 0 \tag{43}$$

is split as a sequence of $\mathbb{Z}_\mathcal{V}[G]$-modules.

(iii) *Assume $\mathcal{O}_{F,\mathrm{tor}}^{\times,T}$ vanishes. Fix an intermediate field F' of F/K, set $\Gamma := \mathrm{Gal}(F'/K)$ and write q_Γ for the functor $\mathbb{Z}_\mathcal{V}[\Gamma] \otimes_{\mathbb{Z}_\mathcal{V}[G]} -$. Then there are natural isomorphisms of Γ-modules*

$$q_\Gamma((\mathrm{Cl}_S^T(F)^\vee)^\Sigma) \cong (\mathrm{Cl}_S^T(F')^\vee)^\Sigma, \quad q_\Gamma(\mathcal{S}_S^T(F)) \cong \mathcal{S}_S^T(F') \quad \textit{and} \quad q_\Gamma(\mathcal{S}_S^T(F)_\Sigma) \cong \mathcal{S}_S^T(F')_\Sigma.$$

In particular, one can fix a splitting of (43) so that, with respect to these isomorphisms, it induces any given splitting of the corresponding exact sequence (43) with F replaced by F'.

Proof. The exact sequence in claim (i) follows directly by applying the functor $-\otimes_\mathbb{Z} \mathbb{Z}_\mathcal{V}$ to the sequence given in [5, Prop. 2.2] and the fact that $(\mathrm{Cl}_S^T(F)^\vee)^\Sigma = (\mathrm{Cl}_S^T(F)^\Sigma)^\vee$.

To prove claim (ii) it is enough to prove the vanishing of the group

$$\mathrm{Ext}^1_{\mathbb{Z}_\mathcal{V}[G]}\big(\mathcal{S}_S^T(F)_\Sigma, (\mathrm{Cl}_S^T(F)^\vee)^\Sigma\big) = \bigoplus_\ell \mathrm{Ext}^1_{\mathbb{Z}_\ell[G]}\big(\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} \mathcal{S}_S^T(F)_\Sigma, \mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} (\mathrm{Cl}_S^T(F)^\vee)^\Sigma\big),$$

where ℓ runs over all primes. This is true since if $\ell \in \Sigma$, then $\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} (\mathrm{Cl}_S^T(F)^\vee)^\Sigma$ vanishes, whilst if $\ell \notin \Sigma$, then the exact sequence in claim (i) implies that $\mathbb{Z}_\ell \otimes_{\mathbb{Z}_\mathcal{V}} \mathcal{S}_S^T(F)_\Sigma$ is a lattice, and hence projective, over $\mathbb{Z}_\ell[G]$ (which is a product of discrete valuation rings in this case since ℓ does not divide $|G|$).

The first displayed isomorphism in claim (iii) exists since $(\mathrm{Cl}_S^T(F)^\vee)^\Sigma$ is finite and of order prime to $|G|$ and so there are canonical isomorphisms

$$q_\Gamma((\mathrm{Cl}_S^T(F)^\vee)^\Sigma) \cong ((\mathrm{Cl}_S^T(F)^\Sigma)^{\mathrm{Gal}(F/F')})^\vee = (\mathrm{Cl}_S^T(F')^\Sigma)^\vee = (\mathrm{Cl}_S^T(F')^\vee)^\Sigma. \tag{44}$$

To derive the second displayed isomorphism we note that, if $\mathcal{O}_{F,\mathrm{tor}}^{\times,T}$ vanishes, then the complex $C_{F,S,T} := \mathsf{R}\Gamma_{c,T}((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathcal{V}}$ defined by Kurihara, Sano and the first named author in [5] is acyclic in degrees greater than one and has cohomology in degree one equal to $\mathcal{S}_S^T(F)$ (by [5, Prop. 2.4(iii)]). In addition, the commutative diagram of exact triangles in [5, Prop. 2.4(i)] combines with the natural isomorphism $\mathbb{Z}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}[G]} \mathsf{R}\Gamma_c((\mathcal{O}_{F,S})_{\text{ét}}, \mathbb{Z}) \cong \mathsf{R}\Gamma_c((\mathcal{O}_{F',S})_{\text{ét}}, \mathbb{Z})$ to induce a canonical isomorphism

$$\epsilon : \mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} C_{F,S,T} \cong C_{F',S,T} \tag{45}$$

in $\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[\Gamma])$. The map $H^1(\epsilon)$ is an isomorphism $q_\Gamma(\mathcal{S}_S^T(F)) \cong \mathcal{S}_S^T(F')$ of the required sort. The third displayed isomorphism now follows immediately from the first two displayed isomorphisms and the fact that the sequence (43) splits.

These isomorphisms then combine to give a canonical map

$$Q_\Gamma : \mathrm{Hom}_{\mathbb{Z}_{\mathcal{V}}[G]}\big(\mathcal{S}_S^T(F), (\mathrm{Cl}_S^T(F)^\vee)^\Sigma\big) \xrightarrow{\theta \mapsto q_\Gamma(\theta)} \mathrm{Hom}_{\mathbb{Z}_{\mathcal{V}}[\Gamma]}\big(\mathcal{S}_S^T(F'), (\mathrm{Cl}_S^T(F')^\vee)^\Sigma\big)$$

and, since these groups are finite and of order prime to $|G|$, this map is surjective. Hence, if we fix a splitting $\sigma_{F'}$ of (43) with F replaced by F', then we can choose a pre-image ϕ_1 under Q_Γ of the homomorphism that corresponds to $\sigma_{F'}$. We now fix any splitting σ_F of (43) for F, write ϕ_2 for the corresponding homomorphism $\mathcal{S}_S^T(F) \to (\mathrm{Cl}_S^T(F)^\vee)^\Sigma$ and then, with H denoting $\mathrm{Gal}(F/F')$, we consider the map

$$\phi : \mathcal{S}_S^T(F) \xrightarrow{(e_H\phi_1, (1-e_H)\phi_2)} e_H(\mathrm{Cl}_S^T(F)^\vee)^\Sigma \oplus (1-e_H)(\mathrm{Cl}_S^T(F)^\vee)^\Sigma = (\mathrm{Cl}_S^T(F)^\vee)^\Sigma$$

Here we use the fact that, since the order of $(\mathrm{Cl}_S^T(F)^\vee)^\Sigma$ is prime to $|G|$, the idempotent e_H acts on this module and induces the displayed direct sum decomposition.

In particular, since q_Γ induces an identification of $e_H(\mathrm{Cl}_S^T(F)^\vee)^\Sigma$ with $(\mathrm{Cl}_S^T(F')^\vee)^\Sigma$, the above map ϕ defines a splitting of (43) that has the property stated in claim (iii) with respect to $\sigma_{F'}$. □

Remark 4.3. The exact sequence in Lemma 4.2(i) (with Σ the set of all primes) implies that the isomorphism $q_\Gamma(\mathcal{S}_S^T(F)) \cong \mathcal{S}_S^T(F')$ in Lemma 4.2(iii) induces homomorphisms $\mathrm{Cl}_S^T(F)^\vee \to \mathrm{Cl}_S^T(F')^\vee$ and $(\mathcal{O}_{F,S}^{\times,T})^* \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathcal{V}} \to (\mathcal{O}_{F',S}^{\times,T})^* \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathcal{V}}$. These maps are respectively induced by the Pontryagin dual of the natural inflation map $\mathrm{Cl}_S(F') \to \mathrm{Cl}_S(F)$ and the $\mathbb{Z}$-linear dual of the inclusion $\mathcal{O}_{F',S}^{\times,T} \to \mathcal{O}_{F,S}^{\times,T}$.

4.2.2. *Statement of the main result.* In this section we assume that K is either a totally real number field or a function field. In these respective cases, we use the following notations:

$$\mathcal{V} = \begin{cases} \{2\} \\ \varnothing \end{cases} \quad \text{and} \quad K_\infty = \begin{cases} K_\infty^{\mathrm{cyc}} & \text{if } K \text{ is a totally real number field,} \\ K_\infty^{\mathrm{con}} & \text{if } K \text{ is a function field.} \end{cases}$$

Here K_∞^{cyc} is the cyclotomic $\mathbb{Z}_p$-extension of the totally real field K (which is the unique $\mathbb{Z}_p$-extension of K if it validates Leopoldt's Conjecture) and K_∞^{con} is the constant $\mathbb{Z}_p$-extension of the function field K.

We set $\Gamma := \mathrm{Gal}(K_\infty/K)$ and fix a topological generator of Γ and therefore identifications $\Gamma = \mathbb{Z}_p$ and $\mathrm{Gal}(K_n/K) = \Gamma/\Gamma^{p^n} = \mathbb{Z}/(p^n)$.

We also fix a finite abelian extension E of K, with Galois group G, and set

$$E_\infty := EK_\infty.$$

We assume E is disjoint from K_∞ so that $\mathrm{Gal}(E_\infty/K)$ identifies with $\mathbb{Z}_p \times G$. In a similar way, for each n, the group

$$G_n := \mathrm{Gal}(E_n/K)$$

identifies with $(\mathbb{Z}/(p^n)) \times G$, and so there are induced identifications of the rings $\mathbb{A} := \mathbb{Z}_{\mathcal{V}}[[\mathrm{Gal}(E_\infty/K)]]$ and $\mathbb{A}_n := \mathbb{Z}_{\mathcal{V}}[\mathrm{Gal}(E_n/K)]$ with $R[G]$ and $R_n[G]$ respectively.

Let S be a set of places of K. Then, in the sequel, we shall say that a set Σ of rational primes is 'S-admissible for E_∞' if the R-module $\varprojlim_n \mathrm{Cl}_S(E_n)_\Sigma \cong \prod_{\ell\in\Sigma} \mathrm{Cl}_S(E_\infty)_\ell$ is finitely generated. In the case that S is empty, we abbreviate 'S-admissible' to 'admissible'.

Remark 4.4. Admissible sets of primes arise in the following ways.
(i) If E is a function field, then Theorem 1.2(i) implies that every set Σ is admissible for the field E_∞^{con}.
(ii) If E is an abelian extension of $\mathbb{Q}$, then the main result of Washington [47] implies that any finite set Σ is admissible for E_∞^{cyc} (cf. Remark 1.4). Moreover, a result of Horie [21, Th. 2] implies that $\mathrm{Cl}(E_\infty^{\mathrm{cyc}})_\ell$ is trivial for a set of primes ℓ of analytic density one, and hence that any admissible set for E_∞^{cyc} is contained in an admissible set of density one.
(iii) Let E be an abelian extension of an imaginary quadratic field k in which p splits. Fix a prime ideal $\mathfrak{p}$ of k above p and let k_∞ be the unique $\mathbb{Z}_p$-extension of k unramified outside $\mathfrak{p}$. Then the result [32, Cor.] of Lamplugh implies any finite set Σ of primes that satisfies [32, Hyp. A] is admissible for Ek_∞.
(iv) If E is a number field that validates the generalized Gross-Kuz'min Conjecture (cf. Remark 4.1), then the equivalence (38) implies that $\Sigma = \{p\}$ is $S(p)$-admissible for E_∞^{cyc}. In addition, if E validates Conjecture 1.3, then every set Σ is $S(p)$-admissible for E_∞^{cyc}.

We fix a finite, non-empty set of places S of K that contains $S_\infty(K)$ and all places that ramify in E_∞ and a finite set of places T of K that is disjoint from S.

By an inductive construction on n using Lemma 4.2(iii), we then fix a Σ-restricted Selmer group $\mathcal{S}_S^T(E_n)_\Sigma$ together with an isomorphism $\mathbb{A}_{n-1} \otimes_{\mathbb{A}_n} \mathcal{S}_S^T(E_n)_\Sigma \cong \mathcal{S}_S^T(E_{n-1})_\Sigma$ of $\mathbb{A}_{n-1}$-modules. The associated limit

$$\mathcal{S}_S^T(E_\infty)_\Sigma := \varprojlim_n \mathcal{S}_S^T(E_n)_\Sigma$$

is therefore a pro-discrete $\mathbb{A}$-module.

For each n we also write $\theta_{E_n,S}^T$ for the (S,T)-relative Stickelberger element in $\mathbb{A}_n$ (as discussed, for example, in [5, §5.1]). We recall that such elements interpolate the values at zero of the T-modified S-truncated Dirichlet L-series attached to complex characters of G_n and are compatible with respect to the natural projection maps $\varrho_{n,m} : \mathbb{A}_n \to \mathbb{A}_m$ for $n > m$. In particular, we may define an element of $\mathbb{A}$ by setting

$$\theta_{E_\infty,S}^T := \varprojlim_n \theta_{E_n,S}^T \in \mathbb{A} = \varprojlim_n \mathbb{A}_n,$$

where the limit is taken with respect to the maps $\varrho_{n,m}$ for $n > m$.

We can now state the main result of §4.2. In this result we use the notion of a 'derived cover' of a pro-discrete module in the sense discussed in Appendix A (see, in particular, Definition A.4 and Remark A.5). For convenience, for each $\mathbb{A}$-module M we also define $\mathbb{A}$-modules M^+ and M^- by setting

$$M^{\pm} := \begin{cases} (1 \pm \tau)M, & \text{if } E \text{ is a CM field,} \\ M, & \text{if } E \text{ is a function field,} \end{cases}$$

where, in the first case, τ denotes the element of $\mathrm{Gal}(E_\infty/K)$ induced by complex conjugation. We note, in particular, that

$$\mathbb{A}^{\pm} = \begin{cases} \mathbb{Z}[1/2][[\mathbb{Z}_p \times G]](1 \pm \tau), & \text{if } E \text{ is a CM field,} \\ \mathbb{Z}[[\mathbb{Z}_p \times G]], & \text{if } E \text{ is a function field,} \end{cases}$$

and hence that if E is a CM field, then there is a direct product decomposition $\mathbb{A} = \mathbb{A}^- \times \mathbb{A}^+$.

Theorem 4.5. *Let S be a finite, non-empty set of places of K that contains $S_\infty(K)$ and all places that ramify in E_∞ and let T be a finite set of places of K that is disjoint from S and such that, for every n, the group $\mathcal{O}^{\times,T}_{E_n,S}$ is torsion-free. Let Σ be a set of rational primes that contains p and all prime divisors of $|G|$. Then there exists a derived cover $\mathcal{DS}^T_S(E_\infty)_\Sigma$ of the pro-discrete $\mathbb{A}$-module of the limit Selmer group $\mathcal{S}^T_S(E_\infty)_\Sigma$ that is canonical up to isomorphism and has all of the following properties.*

(i) *There exists a short exact sequence of $\mathbb{A}$-modules*

$$0 \to \varprojlim\nolimits^1_n \mathrm{cok}(\Delta_{E_n,S}) \to \mathcal{DS}^T_S(E_\infty)_\Sigma \to \mathcal{S}^T_S(E_\infty)_\Sigma \to 0. \tag{46}$$

Here $\Delta_{E_n,S}$ is the natural diagonal map $\mathbb{Z}_\mathcal{V} \xrightarrow{\Delta} Y_{E_n,S} \otimes_\mathbb{Z} \mathbb{Z}_\mathcal{V}$ and the derived limit is taken with respect to the morphisms induced by the restriction of places maps $Y_{E_n,S} \to Y_{E_m,S}$ for $n > m$ and is divisible.

(ii) *Set $\mathcal{F}^\Sigma_n := \mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}^T_S(E_n)^\Sigma)$. Then, for each n, one has $\varrho_{n,n-1}(\mathcal{F}^\Sigma_n) \subseteq \mathcal{F}^\Sigma_{n-1}$, and there exists a canonical exact sequence of $\mathbb{A}^-$-modules*

$$0 \to \theta^T_{E_\infty/K,S} \cdot (\varprojlim_n \mathcal{F}^\Sigma_n)^- \to \varprojlim\nolimits_n (\theta^T_{E_n/K,S} \cdot \mathcal{F}^\Sigma_n)^- \xrightarrow{\kappa_1} \varprojlim\nolimits^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S})^- \to 0.$$

(iii) *Assume Σ is S-admissible for E_∞. Then the $\mathbb{A}^-$-module $\mathcal{DS}^T_S(E_\infty)^-_\Sigma$ is both finitely presented and torsion, and there exists a canonical exact sequence of $\mathbb{A}^-$-modules*

$$0 \to \mathrm{Fit}^0_\mathbb{A}\big(\mathcal{DS}^T_S(E_\infty)_\Sigma\big)^- \to \varprojlim\nolimits_n (\theta^T_{E_n/K,S} \cdot \mathcal{F}^\Sigma_n)^- \xrightarrow{\kappa_2} \varprojlim\nolimits^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S})^- \to 0.$$

Remark 4.6. We make several observations about Theorem 4.5.
(i) The derived limit $\varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S})^-$ is divisible, p-torsion-free and usually non-zero (see Lemma 4.9(iv) below). In particular, since the proof of Theorem 4.5(iii) will also show that $\varprojlim_n (\theta^T_{E_n/K,S} \cdot \mathcal{F}^\Sigma_n)^- = \varprojlim_n \mathrm{Fit}^0_{\mathbb{A}_n}\big(\mathcal{S}^T_S(E_n)_\Sigma\big)^-$, the exact sequence in claim (iii) implies that the inclusion $\mathrm{Fit}^0_\mathbb{A}\big(\mathcal{DS}^T_S(E_\infty)_\Sigma\big) \subseteq \varprojlim_n \mathrm{Fit}^0_{\mathbb{A}_n}\big(\mathcal{S}^T_S(E_n)_\Sigma\big)$ is usually strict.
(ii) The maps κ_1 and κ_2 that occur in claims (ii) and (iii) are both defined as connecting homomorphisms arising from limits of short exact sequences, but the precise link between them is not clear. Nevertheless, a comparison of the exact sequences in claims (ii) and (iii)

suggests that the $\mathbb{A}$-ideals $\theta^T_{E_\infty/K,S}\cdot(\varprojlim_n \mathcal{F}^\Sigma_n)^-$ and $\mathrm{Fit}^0_{\mathbb{A}}\big(\mathcal{DS}^T_S(E_\infty)_\Sigma\big)^-$ should be closely related, and it would be interesting to understand this link precisely.
(iii) Assume E is a CM abelian extension of a totally real field K and that E validates Conjecture 1.3. Then, in any such case, the set S in Theorem 4.5 contains $S(p)$ and so the set Σ^{all} of all rational primes is S-admissible for $E^{\mathrm{cyc}}_\infty/E$. It follows that the $\mathbb{A}$-module $\mathcal{DS}^T_S(E^{\mathrm{cyc}}_\infty) := \mathcal{DS}^T_S(E^{\mathrm{cyc}}_\infty)_{\Sigma^{\mathrm{all}}}$ is finitely presented (by Theorem 4.5(ii)) and, in addition, each group $\mathrm{Cl}^T_S(E_n)^{\Sigma^{\mathrm{all}}}$ vanishes and so $\mathcal{F}^{\Sigma^{\mathrm{all}}}_n = \mathbb{A}_n$. As a special case of the problem mentioned at the end of remark (ii), it therefore seems natural to pose the following question.

Question 4.7. *Assume E is a CM abelian extension of a totally real field K. Then, for any sets S and T as above, is it true that $\mathcal{DS}^T_S(E^{\mathrm{cyc}}_\infty)^-$ is a finitely-presented torsion $\mathbb{A}^-$-module for which one has*

$$\theta^T_{E^{\mathrm{cyc}}_\infty/K,S}\cdot\mathbb{A}^- = \mathrm{Fit}^0_{\mathbb{A}^-}\big(\mathcal{DS}^T_S(E^{\mathrm{cyc}}_\infty)^-\big)?$$

It seems reasonable to regard the prediction of an affirmative answer to this question as a tentative 'main conjecture of integral Iwasawa theory' in this setting.

4.2.3. *Auxiliary complexes.* In the arguments to follow, a key role will be played by the complexes that are described in the next result.

Lemma 4.8. *Fix a finite abelian extension of global fields F/K, with $G := \mathrm{Gal}(F/K)$, and a finite non-empty set S of places of K containing both $S_\infty(K)$ and all places that ramify in F. Let Σ be a finite set of rational primes containing all primes that divide $|G|\cdot|F^\times_{\mathrm{tor}}|$. Then, for any finite set of places T of K that is disjoint from S, there exists a complex $C^\Sigma_{F,S,T}$ of $\mathbb{Z}_\mathcal{V}[G]$-modules with the following properties.*

(i) *$C^\Sigma_{F,S,T}$ belongs to $\mathsf{D}^p(\mathbb{Z}_\mathcal{V}[G])$ and is acyclic outside degrees zero, one and two. There are canonical identifications of cohomology groups*

$$H^i(C^\Sigma_{F,S,T}) = \begin{cases}\mathrm{coker}(\Delta_{F,S}) & \text{if } i=0\\ \mathcal{S}^T_S(F)_\Sigma, & \text{if } i=1\\ \mathcal{O}^{\times,T}_{F,\mathrm{tor}}\otimes_\mathbb{Z}\mathbb{Z}_\mathcal{V} & \text{if } i=2.\end{cases}$$

(ii) *If $\mathcal{O}^{\times,T}_{F,\mathrm{tor}}$ vanishes, then for any given intermediate field F' of F/K, with $\Gamma = \mathrm{Gal}(F'/K)$, there exists an isomorphism $\mathbb{Z}_\mathcal{V}[\Gamma]\otimes^{\mathsf{L}}_{\mathbb{Z}_\mathcal{V}[G]} C^\Sigma_{F,S,T}\cong C^\Sigma_{F',S,T}$ in $\mathsf{D}(\mathbb{Z}_\mathcal{V}[\Gamma])$.*

Proof. If $\Sigma=\Sigma^{\mathrm{all}}$, then we take $C_{F,S,T} := C^{\Sigma^{\mathrm{all}}}_{F,S,T}$ to be the complex $\mathsf{R}\Gamma_{c,T}((\mathcal{O}_{F,S})_\mathcal{W},\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Z}_\mathcal{V}$ that occurs in the proof of Lemma 4.2. In this case, the descriptions of cohomology in claim (i) are given in [5, Prop. 2.4(iii)].

To construct $C^\Sigma_{F,S,T}$ for a general set Σ, we recall that for each pair of bounded above complexes $K^\bullet_1$ and $K^\bullet_2$ of $\mathbb{Z}_\mathcal{V}[G]$-modules, there is a spectral sequence

$$E^2_{p,q} = \prod_{i\in\mathbb{Z}}\mathsf{Ext}^p_{\mathbb{Z}_\mathcal{V}[G]}(H^i(K^\bullet_1),H^{q+i}(K^\bullet_2)) \implies H^{p+q}(\mathsf{R}\mathrm{Hom}_{\mathbb{Z}_\mathcal{V}[G]}(K^\bullet_1,K^\bullet_2)), \qquad (47)$$

the construction of which is given by Verdier in [45, III, 4.6.10].

To use this fact, we set $M_F := \mathrm{Cl}^T_S(F)^\vee$ and take $K^\bullet_1 = C_{F,S,T}$ and $K^\bullet_2 = M^\Sigma_F[-1]$. Then, since the given assumption on Σ implies the groups $\mathsf{Ext}^i_{\mathbb{Z}_\mathcal{V}[G]}(H^{i+1}(C_{F,S,T}),M^\Sigma_F)$ vanish for all $i > 0$, the spectral sequence (47) collapses on its second page to imply bijectivity of the map

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[G])}\big(C_{F,S,T}, M_F^{\Sigma}[-1]\big) &\to \mathrm{Hom}_{\mathbb{Z}_{\mathcal{V}}[G]}\big(\mathcal{S}_S^T(F), M_F^{\Sigma}\big)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_{\mathcal{V}}[G]}\big(M_F^{\Sigma}, M_F^{\Sigma}\big) \oplus \mathrm{Hom}_{\mathbb{Z}_{\mathcal{V}}[G]}\big(\mathcal{S}_S^T(F)_{\Sigma}, M_F^{\Sigma}\big), \quad (48)\end{aligned}$$

where the first map is given by $\theta \mapsto H^1(\theta)$ and the second map is the isomorphism induced by some choice of splitting of the exact sequence (43). With respect to this isomorphism, we write $\kappa_\Sigma = \kappa_{F,\Sigma}$ for the unique morphism in $\mathrm{Hom}_{\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[G])}\big(C_{F,S,T}, M_F^{\Sigma}[-1]\big)$ with $H^1(\kappa_\Sigma) = (\mathrm{id}_{M_F^{\Sigma}}, 0)$ and then define $C_{F,S,T}^{\Sigma}$ to be any complex that lies in an exact triangle in $\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[G])$ of the form

$$C_{F,S,T}^{\Sigma} \to C_{F,S,T} \xrightarrow{\kappa_\Sigma} M_F^{\Sigma}[-1] \to C_{F,S,T}^{\Sigma}[1]. \quad (49)$$

Then, since both $C_{F,S,T}$ and $M_F^{\Sigma}[-1]$ belong to $\mathsf{D}^{\mathsf{p}}(\mathbb{Z}_{\mathcal{V}}[G])$ (the former by [5, Prop. 2.4(iv)] and the latter because $|M_F^{\Sigma}|$ is prime to $|G|$), the complex $C_{F,S,T}^{\Sigma}$ belongs to $\mathsf{D}^{\mathsf{p}}(\mathbb{Z}_{\mathcal{V}}[G])$. In addition, the explicit descriptions of $H^1(\kappa_\Sigma)$ and $H^i(C_{F,S,T})$ for each i combine with an analysis of the long exact sequence of cohomology of the above triangle to give the descriptions of the groups $H^i(C_{F,S,T}^{\Sigma})$ in claim (i).

To construct an isomorphism of the form in claim (ii) we must choose the splitting used in (48) in the manner described in Lemma 4.2(iii). To be precise, we assume that a morphism $\kappa_{F',\Sigma}$ and complex $C_{F',S,T}^{\Sigma}$ have been constructed by the above method with respect to a fixed splitting of the exact sequence (43) with F replaced by F'. We then fix a compatible choice of splitting of (43) in the sense of Lemma 4.2(iii) and use it to construct a morphism $\kappa_{F,\Sigma}$ and an associated complex $C_{F,S,T}^{\Sigma}$. We now consider the following diagram of exact triangles in $\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[\Gamma])$

$$\begin{array}{ccccccc}\mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} C_{F,S,T}^{\Sigma} & \longrightarrow & \mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} C_{F,S,T} & \xrightarrow{1\otimes\kappa_{F,\Sigma}} & \mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} M_F^{\Sigma}[-1] & \longrightarrow & \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle(45)}\big\downarrow & & {\scriptstyle\cong}\big\downarrow & & \\ C_{F',S,T}^{\Sigma} & \longrightarrow & C_{F',S,T} & \xrightarrow{\kappa_{F',\Sigma}} & M_{F'}^{\Sigma}[-1] & \longrightarrow & .\end{array}$$

Here the upper row is the exact triangle obtained by applying the functor $\mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} -$ to the triangle (49). Further, the third vertical arrow is the quasi-isomorphism that is induced by (44) and the quasi-isomorphism

$$\mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} M_F^{\Sigma}[-1] \cong \big(q_\Gamma(M_F^{\Sigma})\big)[-1]$$

that exists since the $\mathbb{Z}_{\mathcal{V}}[G]$-module M_F^{Σ} either vanishes or has projective dimension one.

Then, by explicit computation, one checks that the above construction of $\kappa_{F,\Sigma}$ ensures that the second square in the above diagram commutes after passing to cohomology in degree one. Since the spectral sequence (47) implies bijectivity of the map

$$\mathrm{Hom}_{\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[\Gamma])}\big(\mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} C_{F,S,T}, M_{F'}^{\Sigma}[-1]\big) \xrightarrow{\theta\mapsto H^1(\theta)} \mathrm{Hom}_{\Gamma}\big(H^1(\mathbb{Z}_{\mathcal{V}}[\Gamma] \otimes^{\mathsf{L}}_{\mathbb{Z}_{\mathcal{V}}[G]} C_{F,S,T}), M_{F'}^{\Sigma}\big),$$

this square therefore commutes in $\mathsf{D}(\mathbb{Z}_{\mathcal{V}}[\Gamma])$ and so the diagram can be completed to give a quasi-isomorphism α of the required sort. □

4.2.4. *The proof of Theorem 4.5.* Throughout this section we assume that, for every n, the group $\mathcal{O}^{\times,T}_{E_n,S}$ is torsion-free, and we fix a set of rational primes Σ that is S-admissible for E_∞ and contains both p and all prime divisors of $|G|$.

As a first step, we use Lemma 4.8(ii) (and an inductive construction on n) to fix, for each $n > 1$, a complex $C_n := C^{\Sigma}_{E_n,S,T}$ in $\mathsf{D}^{\mathrm{p}}(\mathbb{A}_n)$ for which there exists an isomorphism $\gamma_n : \mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} C_n \cong C_{n-1}$ in $\mathsf{D}^{\mathrm{p}}(\mathbb{A}_{n-1})$. This family $(C_n, \gamma_n)_n$ satisfies Hypothesis A.1 and so we can set

$$\mathcal{DS}^T_S(E_\infty)_\Sigma := H^1(\mathsf{R}\varprojlim_n C^{\Sigma}_{E_n,S,T}).$$

With this definition, the exact sequence in Theorem 4.5(i) follows directly from Proposition A.3(i) and the result of Lemma 4.8(i) (with F equal to E_n for each n). In addition, the divisibility of $\varprojlim^1_n \mathrm{cok}(\Delta_{E_n,S})$ follows directly from Lemma 3.6 and the fact that each group $\mathrm{cok}(\Delta_{E_n,S})$ is finitely generated as a $\mathbb{Z}_{\mathcal{V}}$-module.

The assertions in Theorem 4.5(ii) are proved in the following result.

Lemma 4.9. *The family* $(\mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}^T_S(E_n)^\Sigma), \varrho_{n,n-1})_{n\in\mathbb{N}}$ *defines an inverse system and the associated ideal* $\mathcal{F} := \varprojlim_m \mathrm{Fit}^0_{\mathbb{A}_m}(\mathrm{Cl}^T_S(E_m)^\Sigma)$ *of* $\mathbb{A}$ *has the following properties.*

(i) *$\mathcal{F}$ is a finitely generated pro-discrete $\mathbb{A}$-module.*

(ii) *For every m, the natural map $\mathcal{F}_{(m)} \to \varrho_m(\mathcal{F}) = \mathrm{Fit}^0_{\mathbb{A}_m}(\mathrm{Cl}^T_S(E_m)^\Sigma)$ is bijective. In particular, the $\mathbb{A}_m$-module $\mathcal{F}_{(m)}$ is invertible.*

(iii) *There exists a canonical exact sequence*

$$0 \to \theta^T_{E_\infty/K,S} \cdot \mathcal{F}^- \to \varprojlim_n \big(\theta^T_{E_n/K,S} \cdot \mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}^T_S(E_n)^\Sigma)\big)^- \to \varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S})^- \to 0.$$

(iv) *The module $\varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S})^-$ is divisible, p-torsion-free and usually non-zero.*

Proof. Since Σ contains p and all prime divisors of $|G|$, the $\mathbb{A}_n$-ideal

$$\mathcal{F}(n) := \mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}^T_S(E_n)^\Sigma)$$

is invertible, so that $\mu_{\mathbb{A}_n}(\mathcal{F}(n)) \leq 2$ (by Remark 2.16), and also such that $\mathcal{F}(n)_p = \mathbb{A}_{n,p}$. For the same reason, for every $n' > n$ the map $\mathbb{A}_n \otimes_{\mathbb{A}_{n'}} \mathrm{Cl}^T_S(E_{n'})^\Sigma \to \mathrm{Cl}^T_S(E_n)^\Sigma$ induced by norms is bijective. These facts imply that the family $(\mathcal{F}(n), \varrho_{n,n-1})_n$ defines a pro-discrete system, so that $\mathcal{F}$ is a pro-discrete $\mathbb{A}$-module, and hence also, by Theorem 2.11(ii), that $\mathcal{F}$ is finitely generated, as required to prove claim (i).

In the present situation, claim (ii) follows directly from Proposition 2.13(i). To prove claim (iii), we set $N' := \theta^T_{E_\infty/K,S} \cdot \mathcal{F}^-$. Then, since the image of $\theta^T_{E_\infty/K,S}$ in $\mathbb{A}^-$ is a non-zero-divisor, the $\mathbb{A}^-$-ideal N' is a pro-discrete $\mathbb{A}$-module (by claim (i)) and each $\mathbb{A}_{(n)}$-module $N'_{(n)}$ is isomorphic to $\mathcal{F}^-_{(n)}$ and so is an invertible $\mathbb{A}^-_n$-module (by claim (ii)). In addition, since

$$\varrho_n(N') = \varrho_n(\theta^T_{E_n/K,S}) \cdot \varrho_n(\mathcal{F}^-) = \theta^T_{E_n/K,S} \cdot \mathcal{F}(n)^-$$

and $\mathcal{F}(n)^-$ is an invertible $\mathbb{A}^-_n$-ideal, the annihilators of $\varrho_n(N')$ and of $\theta^T_{E_n/K,S}$ in $\mathbb{A}^-_n$ coincide.

Given these facts, the existence of an exact sequence as in claim (iii) follows in the case that K is a function field, respectively a totally real number field, by applying Proposition

3.3(iii) with $N = N'$, respectively by applying the exact functor $\mathbb{A}^- \otimes_{\mathbb{A}} -$ to the short exact sequence obtained from Proposition 3.3(iii) with $N = N' \oplus \mathbb{A}^+$.

To prove the first two assertions of claim (iv) we are reduced, by applying Lemma 3.6 with $A_n := \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S})$, to showing that $\varprojlim_n(\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} A_n)$ vanishes. To show this, it is enough to note (55) below implies A_n is only supported on characters of $\mathrm{Gal}(E_n/K)$ that are trivial on the decomposition subgroup of some ramified prime. Indeed, since the decomposition subgroups in $\mathrm{Gal}(E_\infty/K)$ of ramified primes are open, it follows that for each natural number n and for all large enough $m > n$, the image of A_m inside A_n is contained in $p^{m-n} \cdot \mathbb{A}_n$. This fact immediately implies that $\varprojlim_n(\mathbb{Z}_p \otimes_{\mathbb{Z}_\mathcal{V}} A_n)$ vanishes. The final assertion of claim (iv) can also be deduced from this fact since it combines with the unique p-divisibility of $\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}$ to imply that $\varprojlim_n\big((\mathbb{Z}_p/\mathbb{Z}_\mathcal{V}) \otimes_{\mathbb{Z}_\mathcal{V}} A_n\big)$ vanishes if and only if A_n vanishes for all large enough n. □

In order to prove Theorem 4.5(iii) we show first that the family $(C_n)_n$ satisfies the conditions of Proposition A.3(iv). The validity of conditions (a), (b) and (c) follows directly from Lemma 4.8(i) (with F taken to be each field E_n) and the fact each group $\mathcal{O}^{\times,T}_{E_n,S}$ is torsion-free. To verify condition (d), we must prove that the (pro-discrete) $\mathbb{A}$-module $\mathcal{S}^T_S(E_\infty)_\Sigma$ is finitely generated. By Theorem 2.11(ii) we are therefore reduced to showing that the quantities $\mu_{\mathbb{A}_n}(\mathcal{S}^T_S(E_n)_\Sigma)$ are bounded independently of n. To verify this we write C^*_n for the complex $\mathsf{R}\mathrm{Hom}_{\mathbb{A}_n}(C_n, \mathbb{A}_n)$ in $\mathsf{D}^{\mathrm{p}}(\mathbb{A}_n)$. Then the universal coefficient spectral sequence implies C^*_n is acyclic outside degrees -1 and 0 and that there is a canonical exact sequence

$$0 \to \mathrm{Cl}^T_S(E_n)_\Sigma \to H^0(C^*_n) \to X_{E_n,S} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V} \to 0 \tag{50}$$

and an identification of $H^{-1}(C^*_n)$ with the torsion-free group $\mathcal{O}^{\times,T}_{E_n,S} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V}$. By a standard argument (as in [5, §5.4]), these properties combine to imply C^*_n is isomorphic to a complex of $\mathbb{A}_n$-modules $P_n \to \mathbb{A}^{d_n}_n$ in which P_n is locally-free of rank $d_n := \mu_{\mathbb{A}_n}(H^0(C^*_n))$. Since, upon taking $\mathbb{A}_n$-duals, this implies $\mu_{\mathbb{A}_n}(\mathcal{S}^T_S(E_n)_\Sigma) \le \mu_{\mathbb{A}_n}(\mathrm{Hom}_{\mathbb{A}_n}(P_n, \mathbb{A}_n)) \le d_n + 1$ (where the last inequality follows from Remark 2.16), it is therefore enough to prove the integers d_n are bounded independently of n. In view of the exact sequence (50) we are thus reduced to showing that the quantities $\mu_{\mathbb{A}_n}\big(\mathrm{Cl}^T_S(E_n)_\Sigma\big)$ and $\mu_{\mathbb{A}_n}(X_{E_n,S} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V})$ are each bounded independently of n.

In the first case, this follows from the exact sequence

$$Y_{E_n,T} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V} \to \mathrm{Cl}^T_S(E_n)_\Sigma \to \mathrm{Cl}_S(E_n)_\Sigma \to 0, \tag{51}$$

the fact that no place in T splits completely in E_∞ and the assumption that Σ is S-admissible for E_∞.

In the second case we note that the natural sequence

$$0 \to \varprojlim_{n\in\mathbb{N}} X_{E_n,S\setminus S_\infty(E)} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V} \to \varprojlim_{n\in\mathbb{N}} X_{E_n,S} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V} \to \varprojlim_{n\in\mathbb{N}} Y_{E_n,S_\infty(E)} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V} \to 0 \tag{52}$$

is exact since each $\mathbb{Z}[G_n]$-module $Y_{E_n,S_\infty(E)}$ is free, where we recall that $G_n := \mathrm{Gal}(E_n/K)$. In addition, by fixing an extension to E_∞ of each archimedean place of E, one obtains a compatible family of $\mathbb{Z}[G_n]$-bases of the modules $Y_{E_n,S_\infty(E)}$, thereby showing that the R-module $\varprojlim_n Y_{E_n,S_\infty(E)} \otimes_{\mathbb{Z}} \mathbb{Z}_\mathcal{V}$ is free of rank $|S_\infty(E)|$.

It is thus now enough to note that, since each place in $S \setminus S_\infty(E)$ has an open decomposition subgroup in $\mathrm{Gal}(E_\infty/E)$, there are only finitely many places of E_∞ above $S \setminus S_\infty(E)$ and so the R-module $\varprojlim_n X_{E_n,S\setminus S_\infty(E)} \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathcal{V}}$ is both finitely generated and annihilated by ϖ_m for any sufficiently large m.

At this stage, we can apply Proposition A.3(iv) to the family $(C_n, \gamma_n)_n$ to deduce that the $\mathbb{A}$-module $\mathcal{DS}_S^T(E_\infty)_\Sigma$ is a derived cover of $\mathcal{S}_S^T(E_\infty)_\Sigma$ that is canonical up to isomorphism (cf. Remark A.5), that it is finitely presented (as claimed in Theorem 4.5(iii)) and that it satisfies the hypotheses of Proposition 3.3.

We note also that if E is CM, then each module $Y^-_{E_n,S_\infty(E)}$ vanishes and so the exact sequences (51) and (52) combine with the argument of §4.1.2 to imply that $\mathcal{S}_S^T(E_\infty)^-_\Sigma$ is a torsion $\mathbb{A}^-$-module. To show that $\mathcal{DS}_S^T(E_\infty)^-_\Sigma$ is torsion we are therefore reduced, by the exact sequence (46), to showing that $(\varprojlim^1_n \mathrm{coker}(\Delta_{E_n,S}))^-$ is torsion. To do this we first note that an explicit computation of derived limits shows that $(\varprojlim^1_n \mathrm{coker}(\Delta_{E_n,S}))^- = \varprojlim^1_n \mathrm{coker}(\Delta_{E_n,S})^-$. In addition, since each finite place of E has open decomposition group in $\mathrm{Gal}(E_\infty/E)$, there exists a natural number N such that ϖ_N annihilates $Y_{E_n,S\setminus S_\infty(E)}$ for every n. In particular, since each $Y^-_{E_n,S_\infty(E)}$ vanishes, it follows that every module $\mathrm{coker}(\Delta_{E_n,S})^-$, and hence also their derived limit $\varprojlim^1_n \mathrm{coker}(\Delta_{E_n,S})^-$, is annihilated by ϖ_N, as required.

Now, upon combining claim (ii) of Proposition 3.3 with the displayed isomorphism in Proposition A.3(iv), we can deduce the existence of a canonical short exact sequence

$$0 \to \mathrm{Fit}^0_{\mathbb{A}}(\mathcal{DS}_S^T(E_\infty)_\Sigma) \to \varprojlim_n \mathrm{Fit}^0_{\mathbb{A}_n}(\mathcal{S}_S^T(E_n)_\Sigma) \to \varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S}) \to 0. \tag{53}$$

Here we also use the fact that, for each n, one has

$$\mathrm{Ann}_{\mathbb{A}_n}(\mathrm{Fit}^0_{\mathbb{A}_n}(\mathcal{S}_S^T(E_n)_\Sigma)) = \mathrm{Ann}_{\mathbb{A}_n}(\theta^T_{E_n/K,S}). \tag{54}$$

To show this we note that the exact sequence of Lemma 4.2(i) combines with the Dirichlet regulator map to induce an isomorphism of $\mathbb{C}[G_n]$-modules

$$\mathbb{C} \otimes_{\mathbb{Z}_{\mathcal{V}}} \mathcal{S}_S^T(E_n)_\Sigma \cong \mathbb{C} \otimes_{\mathbb{Z}} (\mathcal{O}^{\times,T}_{E_n,S})^* \cong \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{cok}(\Delta_{E_n,S}).$$

Upon comparing the order of vanishing formula for Artin L-series proved in [41, Chap. I, Prop. 3.4] with the definition of Stickelberger elements, one can therefore deduce, for every ψ in $\mathrm{Hom}(G_n, \mathbb{C}^\times)$, that

$$\begin{aligned} e_\psi(\theta^T_{E_n/K,S}) = 0 &\iff \dim_{\mathbb{C}}\big(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}_{\mathcal{V}}} \mathrm{cok}(\Delta_{E_n,S}))\big) > 0 \\ &\iff \dim_{\mathbb{C}}\big(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}_{\mathcal{V}}} \mathcal{S}_S^T(E_n)_\Sigma)\big) > 0. \end{aligned} \tag{55}$$

The claimed equality (54), and hence the exact sequence (53), now follows directly from these equivalences and the fact that the definition of initial Fitting ideals implies for any finitely generated $\mathbb{A}_n$-module M that

$$e_\psi(\mathbb{C} \otimes_{\mathbb{Z}_{\mathcal{V}}} \mathrm{Fit}^0_{\mathbb{A}_n}(M)) = 0 \iff \dim_{\mathbb{C}}\big(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}_{\mathcal{V}}} M)\big) > 0.$$

In light of sequence (53), the proof of Theorem 4.5 is now reduced to showing that, for every n, one has

$$\theta^T_{E_n/K,S} \cdot \mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}_S^T(E_n)^\Sigma)^- = \mathrm{Fit}^0_{\mathbb{A}_n}\big(\mathcal{S}_S^T(E_n)_\Sigma\big)^-.$$

Thus, since the (split) exact sequence (43) implies that

$$\mathrm{Fit}^0_{\mathbb{A}_n}\big(\mathcal{S}^T_S(E_n)\big) = \mathrm{Fit}^0_{\mathbb{A}_n}\big(\mathcal{S}^T_S(E_n)_\Sigma\big)\cdot \mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}^T_S(E_n)^\Sigma),$$

and $\mathrm{Fit}^0_{\mathbb{A}_n}(\mathrm{Cl}^T_S(E_n)^\Sigma)$ is an invertible ideal of $\mathbb{A}_n$ (by Lemma 4.9(ii)), the proof of Theorem 4.5 is completed by the following result.

Proposition 4.10. *For every n, there is an equality of $\mathbb{A}_n^-$-ideals*

$$\theta^T_{E_n/K,S}\cdot \mathbb{A}_n^- = \mathrm{Fit}^0_{\mathbb{A}_n}\big(\mathcal{S}^T_S(E_n)\big)^-.$$

Proof. If K is a function field, then the claimed equality follows directly upon combining the results of Kurihara, Sano and the first author in [5, Th. 1.5(i) and Prop. 3.4] with the data $(K/k,S,T,V,r)$ taken to be $(E_n/K,S,T,\emptyset,0)$ together with the observations that are made in [5, Rem. 3.3(ii) and Rem. 5.3(i)].

In the rest of the argument we therefore assume that K is totally real and E, and hence also every E_n, is a CM field. In this case, the required result will be obtained by combining computations made by Kurihara in [30] with the recent verification by Dasgupta and Kakde [10, Th. 3.5] of the conjecture [30, Conj. 3.2]. To explain this, we adopt the notation of [30], with the extension K/k in loc. cit. now taken to be E_n/K (for any fixed n). For convenience, for every odd prime ℓ we also set

$$\mathcal{A}_\ell := \mathbb{Z}_\ell\otimes_{\mathbb{Z}[1/2]}\mathbb{A}_n^- = (1-\tau)\mathbb{Z}_\ell[G_n] \quad\text{and}\quad A_\ell := \mathbb{Q}_\ell\otimes_{\mathbb{Z}_\ell}\mathcal{A}_\ell.$$

It is then enough for us to show that for every odd prime ℓ there is an equality of $\mathcal{A}_\ell$-ideals

$$\theta^T_{E_n/K,S}\cdot\mathcal{A}_\ell = \mathrm{Fit}^0_{\mathcal{A}_\ell}\big(\mathcal{S}^T_S(E_n)^-_\ell\big) \tag{56}$$

To verify this we follow [30] and so write M° for the linear dual $M^* = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ of a G_n-module M, endowed with the contragredient action of G_n, and use similar notation for morphisms of G_n-modules. We also fix an odd prime ℓ and recall that in [30, §2.1] Kurihara constructs a homomorphism of G_n-modules

$$\psi_{E_n,S}:\mathfrak{A}_{E_n}\to\mathfrak{B}_{E_n}$$

with the following properties.

- The cokernel of $\psi^\circ_{E_n,S}$ is isomorphic to $\mathcal{S}^T_S(E_n)$.
- The $\mathcal{A}_\ell$-modules $\mathfrak{A}^{\circ,-}_{E_n,\ell}$ and $\mathfrak{B}^{\circ,-}_{E_n,\ell}$ are free of the same finite rank.

The first of these properties follows from [30, Prop. 2.3] and the sentence that follows it, whilst the $\mathbb{A}_n$-module $\mathfrak{B}_{E_n}$, and hence also $\mathfrak{B}^\circ_{E_n}$, is free by its construction and the $\mathcal{A}_\ell$-module $\mathfrak{A}^{\circ,-}_{E_n,\ell}$ is free by [30, Prop. 2.4]. These properties combine to imply $\mathrm{Fit}^0_{\mathcal{A}_\ell}\big(\mathcal{S}^T_S(E_n)^-_\ell\big)$ is equal to the $\mathcal{A}_\ell$-ideal generated by $\det(\psi^\circ_{E_n,S})$ and hence show (56) is true if one has

$$\det(\psi^\circ_{E_n,S})\cdot\mathcal{A}_\ell = \theta^T_{E_n/K,S}\cdot\mathcal{A}_\ell.$$

For each character $\chi: G_n\to\mathbb{Q}_\ell^{c,\times}$ and each element x of A_ℓ we set $x^\chi := xe_\chi\in\mathbb{Q}^c_\ell\otimes_{\mathbb{Q}_\ell}A_\ell$. Then to derive the above equality it is enough to show the existence of $\mathcal{A}_\ell$-bases $\underline{b}$ and $\underline{a}$ of $\mathfrak{B}^{\circ,-}_{E_n,\ell}$ and $\mathfrak{A}^{\circ,-}_{E_n,\ell}$ such that the determinant $\det_{\underline{b},\underline{a}}(\psi^\circ_{E_n,S})$ of $\psi^\circ_{E_n,S}$ with respect to these bases is equal to $\theta^T_{E_n/K,S}$, or equivalently that, for every (totally odd) χ one has

$$\det\nolimits_{\underline{b},\underline{a}}(\psi^\circ_{E_n,S})^\chi = \theta^{T,\chi}_{E_n/K,S}. \tag{57}$$

To show this we follow the argument of [30, Th. 3.6]. We therefore write S' for the fixed finite set of places of K that contains S and satisfies the properties specified in [30, §2.1] (with $k = K$) that is used in the constructions of $\mathfrak{A}_{E_n}$ and $\mathfrak{B}_{E_n}$ and then take $\underline{b}$ to be the canonical $\mathcal{A}_\ell$-basis $(e_v^{\mathfrak{B}})_{v\in S'}$ of $\mathfrak{B}^{\circ,-}_{E_n,\ell}$ that is described at the beginning of [30, §3.2]. We further recall from the discussion just before the statement of [30, Th. 3.3] the canonical finitely generated free A_ℓ-module $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} W^{\circ,-}_{S_\infty}$, its canonical basis $\underline{e} = \{e_v\}_{v\in S'}$ and the canonical homomorphism of A_ℓ-modules

$$\psi^\circ : \mathbb{Q}_\ell \otimes_{\mathbb{Z}} W^{\circ,-}_{S_\infty} \to \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathfrak{A}^{\circ,-}_{E_n,\ell}.$$

Then $\det(\psi^\circ)$ is a non-zero divisor in A_ℓ and so the known validity of [30, Conj. 3.2] implies (via its reinterpretation in [30, (3.3)]) the existence of an $\mathcal{A}_\ell$-basis $\underline{a}$ of $\mathfrak{A}^{\circ,-}_{E_n,\ell}$ with the following property: the determinant $\det_{\underline{e},\underline{a}}(\psi^\circ)$ of ψ° with respect to the bases $\underline{e}$ and $\underline{a}$ is equal to

$$\det_{\underline{e},\underline{a}}(\psi^\circ) = \theta^T_{E_n/K,S_\infty} \cdot \prod_{v\in S'\setminus S_\infty} h_v \tag{58}$$

where the elements h_v of $A_\ell^\times$ are as defined in [30, (2.10)].

To verify (57) we set $E_\chi := E_n^{\ker(\chi)}$, $G_\chi := \mathrm{Gal}(E_\chi/K)$ and $A_\chi := \mathbb{Q}_\ell[G_\chi]$, write S_χ for the set of places of K that ramify in E_χ and σ_v for each $v \in S\setminus S_\chi$ for the Frobenius element of v in G_χ. Then the argument at the end of the proof of [30, Th. 3.6] shows that

$$\det_{\underline{b},\underline{a}}(\psi^\circ_{E_\chi,S_\chi})^\chi = \big(\det_{\underline{e},\underline{a}}(\psi^\circ) \cdot \prod_{v\in S'\setminus S_\infty} h_v^{-1}\big)^\chi. \tag{59}$$

We can therefore now obtain the required equality (57) via the computation

$$\begin{aligned}
\det_{\underline{b},\underline{a}}(\psi^\circ_{E_n,S})^\chi &= \det_{\underline{b},\underline{a}}(\psi^\circ_{E_\chi,S})^\chi \\
&= \det_{\underline{b},\underline{a}}(\psi^\circ_{E_\chi,S_\chi})^\chi \cdot \prod_{v\in S\setminus S_\chi} (1-\chi(\sigma_v)^{-1}) \\
&= \big(\det_{\underline{e},\underline{a}}(\psi^\circ) \cdot \prod_{v\in S'\setminus S_\infty} h_v^{-1}\big)^\chi \cdot \prod_{v\in S\setminus S_\chi} (1-\chi(\sigma_v)^{-1}) \\
&= \theta^{T,\chi}_{E_n/K,S_\infty} \cdot \prod_{v\in S\setminus S_\chi} (1-\chi(\sigma_v)^{-1}) \\
&= \theta^{T,\chi}_{E_n/K,S}.
\end{aligned}$$

Here the first equality is true because $A_\chi \otimes_{A_\ell} \psi^\circ_{E_n,S} = \psi^\circ_{E_\chi,S}$, the second is clear if χ is trivial on the decomposition subgroup of any place in $S \setminus S_\chi$ (since then both sides of the stated equality vanish) and otherwise follows from the calculation made towards the end of the proof of [30, Prop. 3.5], the third follows from (59), the fourth from (58) and the final equality directly from the definition of truncated Stickelberger elements. □

4.3. **Inverse limits of unit groups.** In this final section, we assume that $\mathcal{V}$ is empty and show that our methods can also be used to shed light on the structures of inverse limits arising from unit groups.

We start by observing that the results in §4.2 have the following concrete consequence.

Proposition 4.11. *Fix a set of data $F/K, S$ and T as in Lemma 4.8. Let F_∞ be a $\mathbb{Z}_p$-extension of F in which no finite place in S splits completely and for which there exists an admissible set of rational primes Σ. Then the R-module $\varprojlim_n \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^{\times,T}_{F_n,S}, \mathbb{Z})$ is finitely generated, where the transition morphisms are induced by the duals of the inclusions $\mathcal{O}^{\times,T}_{F_n,S} \to \mathcal{O}^{\times,T}_{F_{n+1},S}$.*

Proof. Under the given hypotheses, the argument of §4.2.4 shows that the limit dual Selmer group $\varprojlim_n \mathcal{S}^T_S(F_n)_\Sigma$ is finitely generated over R. In addition, by passing to the limit over n of the exact sequence in Lemma 4.2(i) with F taken to be F_n (and taking account of Remark 4.3), one obtains an exact sequence of R-modules

$$\varprojlim_n \mathcal{S}^T_S(F_n)_\Sigma \to \varprojlim_n \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^{\times,T}_{F_n,S}, \mathbb{Z}) \to \varprojlim{}^1_n (\mathrm{Cl}^T_S(F_n)_\Sigma)^\vee,$$

where the transition morphisms in the central term are as stated above. It is thus enough to note that the last term in this sequence vanishes since each module $\mathrm{Cl}^T_S(F_n)_\Sigma$ is finite. □

In the next result we consider analogously the structure of the inverse limits of p-unit groups with respect to norm maps.

Before stating this result, we note that the problem of describing such limits, and the associated groups of universal norms, directly, rather than by passing to pro-p completions and using Iwasawa theory, is mentioned by Kato in [26] and also directly linked to the results of Coleman in [6] (see Remark 4.13(iii) below).

Proposition 4.12. *Fix an odd prime p and a $\mathbb{Z}_p$-extension of number fields F_∞/F in which no p-adic place splits completely. Let T be a finite set of primes that does not contain p and is such that, for every n, the group of T-modified p-units U^T_n in F_n is torsion-free. Then the family $(U^T_n, N_{F_n/F_{n-1}})_{n\in\mathbb{N}}$ forms an inverse system and the associated R-module $\varprojlim_n U^T_n$ is finitely ∞-presented, respectively finitely generated projective if p is not exceptional, provided that, for every m, one has both*

(i) $\mathrm{Cl}_{S(p)}(F_\infty)_p^{\omega_m=0}$ *is finite;*
(ii) $(\varprojlim{}^1_{m\in\mathbb{N}} U^T_n)^{\omega_m=0}[p^\infty]$ *has bounded exponent.*

Proof. In this argument we write S for the set of places of F that are either archimedean or p-adic, set $\Sigma = \{\, p \,\}$ and use the perfect complexes of R_n-modules

$$C^*_n := \mathsf{R}\mathrm{Hom}_{\mathbb{Z}}(C^\Sigma_{F_n,S,T}, \mathbb{Z})$$

that are constructed in §4.2.4. Then for each n there is a morphism of complexes

$$\epsilon_n : C^*_n \to R_{n-1} \otimes^{\mathsf{L}}_{R_n} C^*_n \cong \mathsf{R}\mathrm{Hom}_{\mathbb{Z}}(R_{n-1} \otimes^{\mathsf{L}}_{R_n} C_n, \mathbb{Z}) \cong \mathsf{R}\mathrm{Hom}_{\mathbb{Z}}(C_{n-1}, \mathbb{Z}) = C^*_{n-1}$$

in $\mathsf{D}(R_n)$. Here the first isomorphism follows from the fact that C_n is perfect and the second is induced by the isomorphism of Lemma 4.8(iv).

We note next that, since condition (i) implies Σ is $S(p)$-admissible for F_∞, the argument following (50) shows that the R-module $\varprojlim_{n\in\mathbb{N}} H^0(C^*_n)$, where the transition morphisms ϵ'_n are induced by ϵ_n, is finitely generated.

We may therefore apply Proposition A.3 to the family $(C^*_n, \epsilon_n)_n$ to deduce that the derived limit $C^*_\infty := \mathsf{R}\varprojlim_n C^*_n$ admits a canonical isomorphism $H^{-1}(C^*_\infty) \cong U^T_\infty$ and a representative

$P \to \Pi$ in $\mathsf{D}(R)$ where Π is a free module of finite rank and P is a finitely generated module such that each $P_{(n)}$ is locally-free. The induced exact sequence of R-modules

$$0 \to U^T_\infty \to P \xrightarrow{\psi} \Pi \to H^0(C^*_\infty) \to 0 \tag{60}$$

implies U^T_∞ is finitely generated if and only if $\mathrm{im}(\psi)$ is finitely presented. By Proposition 2.13(iii), this is true if, for each m, the groups $H^0(C^*_\infty)^{\omega_m=0}[p^\infty]$ have bounded exponent.

To check this we note Proposition A.3(i) implies the existence of an exact sequence

$$0 \to \varprojlim^1_{n\in\mathbb{N}} U^T_n \to H^0(C^*_\infty) \to \varprojlim_{n\in\mathbb{N}} H^0(C^*_n) \to 0.$$

Condition (ii) therefore reduces the required claim to verifying that, for every m, the group $(\varprojlim_n H^0(C^*_n))^{\omega_m=0}[p^\infty]$ has bounded exponent. This now follows from condition (i) and the sequence (50).

Under conditions (i) and (ii), the exact sequence (60) therefore implies that the R-module $H^0(C^*_\infty)$ is finitely 2-presented. In particular, if p is not exceptional, so that P is projective (by Theorem 2.11(iii)), then this combines with the fact that R is a $(2,2)$-ring (by Theorem 1.1), to imply that U^T_∞ is a projective R-module, as claimed. □

Remark 4.13. If $F_\infty = F^{\mathrm{cyc}}_\infty$, then condition (i) in Proposition 4.12 asserts the validity of the generalized Gross-Kuz'min Conjecture for each field F_n. In addition, whilst the proof of Proposition 4.12 shows that condition (ii) need not be necessary for the finite generation of $\varprojlim_n U^T_n$, this condition can at least be analysed explicitly, and we make some observations in this direction.

(i) The derived limit $\varprojlim^1_n U^T_n$ is divisible (by Lemma 3.6) but rarely vanishes. In fact, since each group U^T_n is countable, this limit can vanish only if the system $(U^T_n, N_{F_n/F_{n-1}})_n$ is Mittag-Leffler (cf. [16, p. 242, Prop.]) and this cannot happen if, for $m > n$, the Tate-cohomology groups $\widehat{H}^0(\mathrm{Gal}(K_m/K_n), U^T_m) \cong H^2(\mathrm{Gal}(K_m/K_n), U^T_m)$ have unbounded orders as $m \to \infty$ (as follows, in many cases, from the result of Iwasawa in [22, Prop. 3]).

(ii) If F is abelian over $\mathbb{Q}$, then an explicit analysis using cyclotomic units can be used to show that condition (ii) is satisfied if and only if, for every m, one has

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \bigcap\nolimits_{n\geq m} N_{F_n/F_m}(U^T_n) = \bigcap\nolimits_{n\geq m} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_{F_n/F_m}(U^T_n)).$$

This equality requires that, at each level, universal norms are compatible with the passage to pro-p completion. The results of Bertrandias and Payan in [3, §2.2] show that this condition is not always satisfied, but is satisfied if, for example, F has a unique p-adic place and also contains a non-trivial p-th root of unity.

(iii) For each non-negative integer n, write $\mathbb{Q}^p_n$ for the maximal real subfield of $\mathbb{Q}(e^{2\pi i/p^{n+1}})$. If $F = \mathbb{Q}^p_0$ and $F_\infty = F^{\mathrm{cyc}}_\infty$ (so that $F_n = \mathbb{Q}^p_n$), then the cyclotomic elements

$$c^T_n := \delta_T \cdot (1 - e^{2\pi i/p^{n+1}})(1 - e^{-2\pi i/p^{n+1}})$$

form a norm compatible family (as n varies) and also generate a $\mathrm{Gal}(F_n/\mathbb{Q})$ submodule of U^T_n of finite index. These facts allow one to show directly that $\varprojlim^1_n U^T_n$ vanishes. Since the generalized Gross-Kuz'min Conjecture is also valid for every field F_n, the results on the R-module structure of $\varprojlim_n U^T_n$ given in Proposition 4.12 are therefore unconditionally valid. This observation complements the main result of Coleman in [6] which explicitly characterises the R-submodule of $\varprojlim_n U_n^T$ generated by the norm-compatible family $(c_n^T)_n$.

Appendix A. Derived covers of pro-discrete modules

Throughout this section, we fix a rational prime p, a finite set $\mathcal{V}$ of rational primes that does not contain p and a finite abelian group G. For each non-negative integer n, we define group rings $R_n := \mathbb{Z}_{\mathcal{V}}[\mathbb{Z}/(p^n)]$ and $\mathbb{A}_n := R_n[G]$ and then set

$$R := \mathbb{Z}_{\mathcal{V}}[[\mathbb{Z}_p]] = \varprojlim_n R_n \quad \text{and} \quad \mathbb{A} := R[G] = \varprojlim_n \mathbb{A}_n,$$

where in both cases the limits are taken with respect to the natural projection maps.

We shall use derived limits of families of complexes to define a notion of 'derived covers' for pro-discrete $\mathbb{A}$-modules.

As a first step, we review some basic properties of derived limits of complexes. To do this we fix a family of complexes $(X_n^\bullet)_{n\in\mathbb{N}}$ of $\mathbb{A}$-modules satisfying the following hypothesis.

Hypothesis A.1.

(i) *Each $X_n^\bullet$ is a complex of $\mathbb{A}_n$-modules.*

(ii) *For each $n > 1$, there exists an isomorphism in $\mathsf{D}(\mathbb{A}_{n-1})$ of the form*

$$\gamma_n : \mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} X_n^\bullet \cong X_{n-1}^\bullet.$$

We also use γ_n to denote the induced morphism $X_n^\bullet \to X_{n-1}^\bullet$ in $\mathsf{D}(\mathbb{A}_n)$.

Definition A.2. The 'derived limit' of any family $(X_n^\bullet)_{n\in\mathbb{N}}$ of complexes satisfying Hypothesis A.1 is a complex $\mathsf{R}\varprojlim_n X_n^\bullet$ that lies in an exact triangle in $\mathsf{D}(\mathbb{A})$ of the form

$$\mathsf{R}\varprojlim_n X_n^\bullet \to \prod_{n\in\mathbb{N}} X_n^\bullet \xrightarrow{1-\gamma_n} \prod_{n\in\mathbb{N}} X_n^\bullet \to (\mathsf{R}\varprojlim_n X_n^\bullet)[1]. \tag{61}$$

Such a complex is unique up to isomorphism in $\mathsf{D}(\mathbb{A})$.

The following result describes some useful properties of this construction.

Proposition A.3. *Assume that there exist integers a and b with $a \le b$ such that, for every n, the complex $X_n^\bullet$ is acyclic in all degrees less than a and greater than b. Then the following claims are valid.*

(i) $\mathsf{R}\varprojlim_n X_n^\bullet$ is acyclic in all degrees less than a and greater than b. Further, there is a canonical isomorphism $H^a(\mathsf{R}\varprojlim_n X_n^\bullet) \cong \varprojlim_{n\in\mathbb{N}} H^a(X_n^\bullet)$ and, for each $i > a$, a canonical short exact sequence

$$0 \to \varprojlim^1_{n\in\mathbb{N}} H^{i-1}(X_n^\bullet) \to H^i(\mathsf{R}\varprojlim_n X_n^\bullet) \to \varprojlim_{n\in\mathbb{N}} H^i(X_n^\bullet) \to 0,$$

where all limits are taken with respect to the morphisms $H^{i-1}(\gamma_n)$ and $H^i(\gamma_n)$.

(ii) Suppose to be given two further families $(Y_n^\bullet)$ and $(Z_n^\bullet)$ satisfying Hypothesis A.1, together with a collection of compatible exact triangles

$$\Delta_n : X_n^\bullet \to Y_n^\bullet \to Z_n^\bullet \to X_n^\bullet[1]$$

in $\mathsf{D}(\mathbb{A}_n)$*. Then there is an exact triangle in* $\mathsf{D}(\mathbb{A})$

$$\mathsf{R}\varprojlim_{n\in\mathbb{N}} X_n^\bullet \to \mathsf{R}\varprojlim_{n\in\mathbb{N}} Y_n^\bullet \to \mathsf{R}\varprojlim_{n\in\mathbb{N}} Z_n^\bullet \to (\mathsf{R}\varprojlim_{n\in\mathbb{N}} X_n^\bullet)[1].$$

(iii) For each m *there exists a canonical morphism* $\mathbb{A}_m \otimes^{\mathsf{L}}_{\mathbb{A}} \mathsf{R}\varprojlim_n X_n^\bullet \to X_m^\bullet$ *in* $\mathsf{D}(\mathbb{A}_m)$*.*

(iv) Assume that $b = a+1$ *and that the following conditions are satisfied:*

(a) *for all* n *the complex* $X_n^\bullet$ *belongs to* $\mathsf{D}^{\mathrm{p}}(\mathbb{A}_n)$*;*
(b) *for all* n *the module* $H^a(X_n^\bullet)$ *is* $\mathbb{Z}_{\mathcal{V}}$*-free;*
(c) *for all* n *the* $\mathbb{Q}\otimes_{\mathbb{Z}_{\mathcal{V}}}\mathbb{A}_n$*-modules spanned by* $H^a(X_n^\bullet)$ *and* $H^{a+1}(X_n^\bullet)$ *are isomorphic.*
(d) $\varprojlim_{n\in\mathbb{N}} H^{a+1}(X_n^\bullet)$ *is a finitely generated pro-discrete* $\mathbb{A}$*-module.*

Then $\mathsf{R}\varprojlim_n X_n^\bullet$ *is isomorphic in* $\mathsf{D}(\mathbb{A})$ *to a complex of the form* $P \to \Pi$ *in which* Π *occurs in degree* $a+1$ *and is free of finite rank, and* P *identifies with* $\varprojlim_n P_{(n)}$ *and is such that each* $\mathbb{A}_n$*-module* $P_{(n)}$ *is locally-free of the same rank as* Π*. Moreover, for each* m *the morphism in claim (iii) is an isomorphism and the induced homomorphism of* $\mathbb{A}_m$*-modules*

$$H^{a+1}(\mathsf{R}\varprojlim_{n\in\mathbb{N}} X_n^\bullet)_{(m)} \to H^{a+1}(X_m^\bullet)$$

is bijective.

Proof. The long exact sequence of cohomology of the triangle (61) degenerates to give in each degree k a short exact sequence

$$0 \to \varprojlim^1_{n\in\mathbb{N}} H^{k-1}(X_n^\bullet) \to H^k(\mathsf{R}\varprojlim_{n\in\mathbb{N}} X_n^\bullet) \to \varprojlim_{n\in\mathbb{N}} H^k(X_n^\bullet) \to 0, \tag{62}$$

in which each limit is taken with respect to the transition morphisms induced by the respective maps $H^i(\gamma_n)$. The isomorphism in the second assertion of claim (i), together with the acyclicity of $\mathsf{R}\varprojlim_n X_n^\bullet$ in all degrees less than a and greater than $b+1$, follow directly from these sequences.

The acyclicity in degree $b+1$ follows in the same way since each morphism

$$H^b(\gamma_n) : H^b(X_n^\bullet) \to H^b(X_{n-1}^\bullet)$$

is surjective (as $X_n^\bullet$ is acyclic in degrees greater than b).

To prove claim (ii) we note that restriction of scalars along $\mathbb{A} \to \mathbb{A}_n$ is an exact functor between the respective module categories and hence that each triangle Δ_n is exact in $\mathsf{D}(\mathbb{A})$. In particular, since the category of $\mathbb{A}$-modules has exact products, there exists an exact triangle in $\mathsf{D}(\mathbb{A})$ of the form

$$\prod_{n\in\mathbb{N}} X_n^\bullet \to \prod_{n\in\mathbb{N}} Y_n^\bullet \to \prod_{n\in\mathbb{N}} Z_n^\bullet \to \prod_{n\in\mathbb{N}} X_n^\bullet[1].$$

The claimed triangle then follows by combining this fact with the triangle (61) and then applying the octahedral axiom.

To prove claim (iii), for each $m \geq n$ we write $\gamma_{m,n} : X_m^\bullet \to X_n^\bullet$ for the composite morphism $\gamma_{n+1} \circ \cdots \circ \gamma_{m-1} \circ \gamma_m$. Then we have a commutative square in $\mathsf{D}(\mathbb{A})$

$$\begin{array}{ccc}\mathbb{A}_n \otimes^{\mathsf{L}}_{\mathbb{A}} \prod_{m\in\mathbb{N}} X^\bullet_m & \xrightarrow{1-\gamma_m} & \mathbb{A}_n \otimes^{\mathsf{L}}_{\mathbb{A}} \prod_{m\in\mathbb{N}} X^\bullet_m \\ \Big\downarrow{\scriptstyle (\gamma_{m,n})_{m\ge n}} & & \Big\downarrow{\scriptstyle (\gamma_{m,n})_{m\ge n}} \\ \prod_{m\ge n} X^\bullet_n & \xrightarrow{1-\mathrm{id}} & \prod_{m\ge n} X^\bullet_n.\end{array}$$

This diagram combines with the triangle (61) to induce a morphism from $\mathbb{A}_n \otimes^{\mathsf{L}}_{\mathbb{A}} \mathsf{R}\varprojlim_{m\in\mathbb{N}} X^\bullet_m$ to the derived limit $\mathsf{R}\varprojlim_{m\in\mathbb{N}} X^\bullet_n$ of the constant system $(X^\bullet_n)_{m\in\mathbb{N}}$. Since this latter limit is canonically isomorphic in $\mathsf{D}(\mathbb{A})$ to $X^\bullet_n$ we thus obtain the desired morphism of complexes.

To prove claim (iv) we note first that the stated condition (d) allows us to fix a free $\mathbb{A}$-module Π of finite rank t, together with a surjective homomorphism

$$\pi : \Pi \to \varprojlim_{n\in\mathbb{N}} H^{a+1}(X^\bullet_n).$$

For each n we write Π_n for the free, rank t, $\mathbb{A}_n$-module $\mathbb{A}_n \otimes_{\mathbb{A}} \Pi$ and we note that, since $\varprojlim_{n\in\mathbb{N}} H^{a+1}(X^\bullet_n)$ is assumed to be pro-discrete, the map $\pi_n : \Pi_n \to H^{a+1}(X^\bullet_n)$ induced by π is surjective (cf. Remark 2.7).

Then, by the same argument as used just after (50), the stated conditions (a), (b) and (c) combine to imply that $X^\bullet_n$ is isomorphic in $\mathsf{D}(\mathbb{A}_n)$ to a complex $\Pi^\bullet_n$ of the form $P_n \xrightarrow{\phi_n} \Pi_n$, where P_n is a finitely generated locally-free (and hence projective) $\mathbb{A}_n$-module of rank t that is placed in degree a, and $\mathrm{cok}(\phi_n)$ is identified with $H^{a+1}(X^\bullet_n)$ via the map π_n.

The composite morphism

$$\Pi^\bullet_n \cong X^\bullet_n \xrightarrow{\gamma_n} X^\bullet_{n-1} \cong \Pi^\bullet_{n-1}$$

in $\mathsf{D}(\mathbb{A}_n)$ can then be represented by a concrete morphism of complexes $\theta^\bullet_n = (\theta^a_n, \theta^{a+1}_n)$ and, by changing $\theta^\bullet_n$ by a homotopy (if necessary), we can assume that θ^{a+1}_n is the natural projection map $\rho_n : \Pi_n \to \Pi_{n-1}$.

Now, as θ^a_n represents γ_n, the induced morphism of complexes

$$\mathbb{A}_{n-1} \otimes_{\mathbb{A}_n} \theta^\bullet_n : \mathbb{A}_{n-1} \otimes_{\mathbb{A}_n} \Pi^\bullet_n \to \Pi^\bullet_{n-1}$$

is a quasi-isomorphism. In particular, since the map $\mathbb{A}_{n-1} \otimes_{\mathbb{A}_n} \Pi_n \to \Pi_{n-1}$ that is induced by $\theta^{a+1}_n = \rho_n$ is bijective, the map $\mathbb{A}_{n-1} \otimes_{\mathbb{A}_n} P_n \to P_{n-1}$ induced by θ^{a+1}_n must also be bijective. The family $(P_n, \theta^a_n)_n$ is therefore a pro-discrete system and so Theorem 2.11(ii) implies that its limit $P := \varprojlim_{n\in\mathbb{N}} P_n$ is a finitely generated (pro-discrete) $\mathbb{A}$-module. From Proposition 2.13(i) we also know that the canonical map $P_{(n)} \to P_n$ is bijective for every n.

Writing ϕ_∞ for the limit homomorphism $\varprojlim_n \phi_n$, we next claim that the complex $\Pi^\bullet$ given by $P \xrightarrow{\phi_\infty} \Pi$, where P is placed in degree a, is isomorphic to $\mathsf{R}\varprojlim_{n\in\mathbb{N}} X^\bullet_n$ in $\mathsf{D}(\mathbb{A})$.

To show this we note that, since $\theta^\bullet_n$ represents γ_n in $\mathsf{D}(\mathbb{A}_n)$, the derived limit $\mathsf{R}\varprojlim_{n\in\mathbb{N}} \Pi^\bullet_n$ of the family $(\Pi^\bullet_n, \theta^\bullet_n)$ is isomorphic in $\mathsf{D}(\mathbb{A})$ to $\mathsf{R}\varprojlim_{n\in\mathbb{N}} X^\bullet_n$. It is therefore enough to show that $\Pi^\bullet$ is isomorphic in $\mathsf{D}(\mathbb{A})$ to $\mathsf{R}\varprojlim_{n\in\mathbb{N}} \Pi^\bullet_n$.

Write $\mathsf{Mod}^{\mathbb{N}}(\mathbb{A})$ for the abelian category of inverse systems (M_n, ψ_n) of $\mathbb{A}$-modules where each M_n is naturally a $\mathbb{A}_n$-module and $\psi_n : M_n \to M_{n-1}$ is a homomorphism of $\mathbb{A}$-modules.

Then, since the category $\mathsf{Mod}^{\mathbb{N}}(\mathbb{A})$ has enough injectives, the natural inverse limit functor $\varprojlim_{n\in\mathbb{N}}: \mathsf{Mod}^{\mathbb{N}}(\mathbb{A}) \to \mathsf{Mod}(\mathbb{A})$ induces a functor $\mathsf{R}\varprojlim'_{n\in\mathbb{N}}: \mathsf{D}(\mathsf{Mod}^{\mathbb{N}}(\mathbb{A})) \to \mathsf{D}(\mathbb{A})$ on the corresponding derived categories.

The key point now is that the family of complexes $(\Pi_n^\bullet)_n$ can be regarded as an object of $\mathsf{D}(\mathsf{Mod}^{\mathbb{N}}(\mathbb{A}))$. In particular, since both of the maps θ_n^a and θ_n^{a+1} are surjective, a straightforward calculation (cf. [38, Lem. 0CQD]) shows that $\mathsf{R}\varprojlim_{n\in\mathbb{N}}\Pi_n^\bullet$ is isomorphic in $\mathsf{D}(\mathbb{A})$ to the complex $\mathsf{R}\varprojlim'_{n\in\mathbb{N}}\Pi_n^\bullet \cong \Pi^\bullet$, as required. This completes the proof of the first assertion of claim (iv).

The final assertion of claim (iv) is then true since, in this case, the morphism in claim (iii) coincides with the composite morphism

$$\mathsf{R}\varprojlim_{n\in\mathbb{N}} X_n^\bullet \cong \Pi^\bullet \xrightarrow{x\mapsto 1\otimes x} \mathbb{A}_m\otimes_{\mathbb{A}}\Pi^\bullet = \Pi_m^\bullet \cong X_m^\bullet,$$

and so the induced map $H^{a+1}(\mathsf{R}\varprojlim_{n\in\mathbb{N}} X_n^\bullet)_{(m)} \to H^{a+1}(X_m^\bullet)$ is bijective since $\operatorname{cok}(\phi_\infty)_{(m)}$ identifies with $\operatorname{cok}(\phi_m)$. □

Definition A.4. Let M be a pro-discrete $\mathbb{A}$-module corresponding to an inverse system $(M_n,\pi_n)_{n\in\mathbb{N}}$. Assume that, for each $n>1$, there exists a commutative diagram

$$\begin{array}{ccccccc} P_n^{-1} & \xrightarrow{\phi_n} & P_n^0 & \longrightarrow & M_n & \longrightarrow & 0\\ \downarrow{\scriptstyle \pi_n^{-1}} & & \downarrow{\scriptstyle \pi_n^0} & & \downarrow{\scriptstyle \pi_n} & & \\ P_{n-1}^{-1} & \xrightarrow{\phi_{n-1}} & P_{n-1}^0 & \longrightarrow & M_{n-1} & \longrightarrow & 0 \end{array}$$

of projective presentations of the $\mathbb{A}_n$-modules M_n, with the property that π_n^a induces an isomorphism $\mathbb{A}_{n-1}\otimes_{\mathbb{A}_n}P_n^a \cong P_{n-1}^a$ for $a\in\{-1,0\}$. Writing $P_n^\bullet$ for the complex $P_n^{-1}\xrightarrow{\phi_n}P_n^0$ in $\mathsf{D}(\mathbb{A}_n)$, where the first term is placed in degree -1, and $\pi_n^\bullet$ for the morphism $P_n^\bullet\to P_{n-1}^\bullet$ represented by the pair (π_n^{-1},π_n^0), the result of Proposition A.3(i) implies the existence of an exact sequence of $\mathbb{A}$-modules

$$0\to \varprojlim_n^1 \ker(\phi_n) \to H^0(\mathsf{R}\varprojlim_n P_n^\bullet) \to M \to 0, \tag{63}$$

where the respective limits are taken with respect to the morphisms π_n^{-1} and $\pi_n^\bullet$. We refer to any module $H^0(\mathsf{R}\varprojlim_n P_n^\bullet)$ arising in this way as a 'derived cover' of M.

Remark A.5. In general, if one changes the choice of projective resolution $P_n^\bullet$ of M_n, then the corresponding $\mathbb{A}_n$-modules $\ker(\phi_n)$ that occur in (63) are homotopy equivalent in the sense of Jannsen [24] (this follows, for example, from [36, Prop. (5.4.3)]), but need not be isomorphic. However, Proposition A.3(ii) implies that the isomorphism class of a derived cover is independent of choices in the following sense: if $(Q_n^\bullet,\tau_n^\bullet)$ is any compatible family of projective resolutions of the modules M_n for which, for every $n>1$, there exists a commutative diagram in $\mathsf{D}(\mathbb{A}_n)$

$$\begin{array}{ccc} Q_n^\bullet & \xrightarrow{j_n} & P_n^\bullet\\ {\scriptstyle \tau_n^\bullet}\downarrow & & \downarrow{\scriptstyle \pi_n^\bullet}\\ Q_{n-1}^\bullet & \xrightarrow{j_{n-1}} & P_{n-1}^\bullet \end{array}$$

in which each map j_n is an isomorphism, then the derived covers of M that are defined via $(P_n^\bullet, \pi_n^\bullet)$ and $(Q_n^\bullet, \tau_n^\bullet)$ are isomorphic $\mathbb{A}$-modules. The argument of Proposition A.3 shows that this observation applies, in particular, to the pro-discrete modules $M = \varprojlim_{n\in\mathbb{N}} H^{a+1}(X_n^\bullet)$ that occur in the setting of claim (iv).

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